GENERATING INTEREST RATE SCENARIOS FOR FIXED INCOME PORTFOLIO OPTIMISATION

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ABSTRACT
One of the main sources of uncertainty in analysing risk and return properties of a portfolio of fixed income securities is the stochastic evolution of the shape of the term structure of interest rates. We estimate a model that fits the South African term structure of interest rates, using a Kalman filter approach. Our model includes four latent factors and observable macro-economic variables (capacity utilisation, inflation and repo-rate). Our goal is to capture the dynamic interactions between the macro-economy and the term structure in such a way that the resulting model can be used to generate interest rate scenario trees that are suitable for fixed income portfolio optimisation. An important input into our scenario generator is the investor's view on the future evolution of the repo-rate.

In this paper we will provide details on our model and report on the results of the estimation and scenario generation.

KEYWORDS
Term structure; yield curve; Kalman filter; macro-economic; scenario generation; Nelson-Siegel curve; Svensson curve

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1. INTRODUCTION

One of the main sources of uncertainty in analysing the risk and return properties of a portfolio of fixed income securities is the stochastic evolution of the shape of the term structure of interest rates (or yield curve). Inspired by the research of Diebold et al. (2006) we estimate a model that fits the South African term structure of interest rates, using a Kalman filter approach. Diebold et al. (2006) characterise the yield curve using three latent factors, namely *level*, *slope* and *curvature*. To model the dynamic interactions between the macro-economy and the yield curve, they also included observable macro-economic variables, specifically *real activity*, *inflation* and a *monetary policy instrument*. Other examples where a latent factor model approach is used to characterise the yield curve and that explicitly include macro-economic factors can be found in Ang & Piazzesi (2003), Hördahl et al. (2002) and Wu (2002). These examples, however, only consider a unidirectional linkage between the macro-economy and the yield curve. Kozicki & Tinsley (2001), Dewachter & Lyrio (2002) and Rudebusch & Wu (2003) allow for implicit feedback.

To capture the dynamics of the yield curve, Diebold et al. (2006) do not use a no-arbitrage factor representation such as the typically used affine no-arbitrage models (see for example Duffee, 2002 and Brousseau, 2002) or canonical affine no-arbitrage models (see for example Rudebusch & Wu, 2003). Instead of using a no-arbitrage representation Diebold et al. (2006) suggest using a three-factor term structure model based on the yield curve model of Nelson & Siegel (1987), as used in Diebold & Li (2006), and interpret these factors as *level*, *slope* and *curvature*. Diebold & Li (2006) proposes a two-step procedure to estimate the dynamics of the yield curve. The procedure firstly estimates the three latent factors and secondly estimates an autoregressive model for these factors. Diebold & Li (2006) use these models to forecast the term structure. Diebold et al. (2006) proposed a one-step approach by introducing an integrated state-space modelling approach which is preferred over the two-step Diebold-Li approach. This Kalman filter approach simultaneously fits the yield curve and estimates the underlying dynamics of these factors. The model also incorporates the estimation of the macro-economic factors and the link between the macro-economy and the latent factors driving the yield curve.

In Section 2, we describe the Kalman filter state-space modelling approach for the basic three-factor *yields-only* model used by Diebold et al. (2006). This model uses only three latent factors of the yield curve and does not include macro-economic factors. We will describe the model estimation for the South African term structure and introduce a four-factor model based on the yield curve model of Svensson (1994). The four-factor model is introduced because the Nelson & Siegel (1987) is not flexible enough to get an acceptable fit to the South African term structure.

In Section 3, we incorporate macro-economic variables (*capacity utilisation*, *inflation* and *repo-rate*). Our goal is to capture the dynamic interactions between the macro-economy and the term structure in such a way that the resulting model can be used to generate interest rate scenario trees that are suitable for fixed income portfolio optimisation. Section 4 describes our approach. An important input into our scenario generator is the investor's view on the future evolution of the repo-rate. We also discuss the existence of arbitrage in the scenario trees and propose a method to reduce arbitrage opportunities.

We offer concluding remarks in Section 5.

2. YIELDS-ONLY MODEL

In this section we introduce the factor model representation of the yield curve. Following Diebold et al. (2006), we start with *yields-only model* using the three factor representation of Nelson-Siegel and use this as a benchmark for the four factor representation of Svensson. By using the more flexible four-factor model, we obtain a better fit. Since all the models that are described in this section are fitted using a Kalman filter approach, we start this section with an overview of the Kalman filter.

2.1 THE KALMAN FILTER

The Kalman filter, introduced by Kalman (1960), is a popular technique used in signal processing, control engineering and other fields. The main idea is to represent a dynamic system in terms of states (the unobserved underlying Markov process). The *state equation* (or *transition equation*) describes the dynamics of this process while the *observation equation* (or *measurement equation*) relates the observables with the unobserved states. The advantage of using a state-space representation (defined below) is that it allows the modeller to include information on the evolution of the yield curve over time as well as information contained in the cross section of interest rates at each point in time.

Following Hamilton (Hamilton: Chap 13, 1994), let y_t denote a vector of variables (yields in our case) observed at date t that can be described in terms of f_t, a vector of unobservable states. The *state-space representation* of the dynamics of *y* is then given the following system of equations:

$$f_t = Af_{t-1} + \eta_t \qquad \text{[Transition equation]}$$

$$y_t = Bx_t + \Lambda f_t + \varepsilon_t \qquad \text{[Measurement equation]}$$

where the matrices *A*, *B* and Λ have appropriate dimensions, x_t is a vector of exogenous variables. The disturbances η_t and ε_t are vector white noise:

$$E\left(\eta_t \eta_\tau'\right) = \begin{cases} Q & \text{for } t = \tau \\ 0 & \text{otherwise} \end{cases},$$

$$E\left(\varepsilon_t \varepsilon_\tau'\right) = \begin{cases} H & \text{for } t = \tau \\ 0 & \text{otherwise} \end{cases},$$

where the matrices Q and H have appropriate dimensions. The disturbances η_t and ε_t are also assumed to be uncorrelated at all lags:

$$E\left(\eta_t \varepsilon_\tau\right)=0, \quad \text{for all } t \text{ and } \tau.,$$

The Kalman filter is a sequential algorithm that calculates the best predictor of the unobserved states, given all previous observations. The details will be given later.

2.2 FACTOR REPRESENTATION

The main aim of the factor model approach is to represent the term structure (a large set of yields with various maturities) as a function of a smaller set of unobservable factors. The Nelson-Siegel representation (Nelson & Siegel, 1987) produces reliable and reasonable estimation results and has become one of the popular approaches adopted by central banks for yield curve estimation (Bank of International Settlements, 1999). The Nelson-Siegel model, derived from a parametric functional form for the forward rates, uses only four parameters to define a more parsimonious and stable representation of the whole term structure:

$$y(\tau)=\beta_1+\beta_2\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right)+\beta_3\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}-e^{-\lambda\tau}\right),$$

where $y(\tau)$ is the zero coupon yield with maturity τ and β_1, β_2, β_3 and λ are the model parameters. As demonstrated by Diebold & Li (2006), the latent factors β_1, β_2 and β_3 of the Nelson-Siegel representation of the yield curve, can be interpreted as level, slope and curvature and the terms that multiply these factors are factor loadings. The parameter λ determines the shape of the curve and does not have a direct economic interpretation.

Diebold & Li (2006) rewrite the representation as

$$y_t(\tau)=L_t+S_t\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right)+C_t\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}-e^{-\lambda\tau}\right)$$

where L_t, S_t and C_t are the, time-varying unobserved factors, β_1, β_2 and β_3.

Diebold et al. (2006) describe the state-space system as follow: The dynamics of the unobservable factors, L_t, S_t and C_t, are modelled as a vector autoregressive process of the first order which forms a state-space system. The ARMA state vector dynamics may be of any order, but the VAR(1) assumption is maintained for transparency and parsimony. The dynamics of the state vector is governed by the *transition equation*

$$\begin{pmatrix} L_t - \mu_L \\ S_t - \mu_S \\ C_t - \mu_C \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} L_{t-1} - \mu_L \\ S_{t-1} - \mu_S \\ C_{t-1} - \mu_C \end{pmatrix} + \begin{pmatrix} \eta_t(L) \\ \eta_t(S) \\ \eta_t(C) \end{pmatrix},$$

where $t = 1,...,T$. By fixing the parameter λ (to be specified later), the *measurement equation*, which relates a set of N yields of the yield curve, with maturities $\tau_1,...,\tau_N$, to the three unobserved factors, are

$$\begin{pmatrix} y_t(\tau_1) \\ y_t(\tau_2) \\ \vdots \\ y_t(\tau_N) \end{pmatrix} = \begin{pmatrix} 1 & \frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1} & \frac{1-e^{-\lambda\tau_1}}{\lambda\tau_1} - e^{-\lambda\tau_1} \\ 1 & \frac{1-e^{-\lambda\tau_2}}{\lambda\tau_2} & \frac{1-e^{-\lambda\tau_2}}{\lambda\tau_2} - e^{-\lambda\tau_2} \\ \vdots & \vdots & \vdots \\ 1 & \frac{1-e^{-\lambda\tau_N}}{\lambda\tau_3} & \frac{1-e^{-\lambda\tau_N}}{\lambda\tau_N} - e^{-\lambda\tau_N} \end{pmatrix} \begin{pmatrix} L_t \\ S_t \\ C_t \end{pmatrix} + \begin{pmatrix} \varepsilon_t(\tau_1) \\ \varepsilon_t(\tau_2) \\ \vdots \\ \varepsilon_t(\tau_N) \end{pmatrix},$$

where $t = 1,...,T$. The state-space system can be written in matrix notation as

$$(f_t - \mu) = A(f_{t-1} - \mu) + \eta_t,$$

$$y_t = \Lambda f_t + \varepsilon_t.$$

The white noise disturbances in the transition and measurement equations are required to be orthogonal to each other and to the initial state for the linear least-squares optimality of the Kalman filter:

$$\begin{pmatrix} \eta_t \\ \varepsilon_t \end{pmatrix} \sim WN\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} Q & 0 \\ 0 & H \end{pmatrix}\right],$$

$$E(f_0 \eta_t') = 0,$$

$$E(f_0 \varepsilon_t') = 0.$$

Diebold et al. (2006) assume that the Q matrix is non-diagonal to allow the shocks to the three term structure factors to be correlated. The H matrix is assumed to be diagonal, which implies that the deviations of the yields of various maturities from the yield curve are uncorrelated. This is quite standard and the same as in no-arbitrage term structure models and is required for computational tractability given the large number of observed yields.

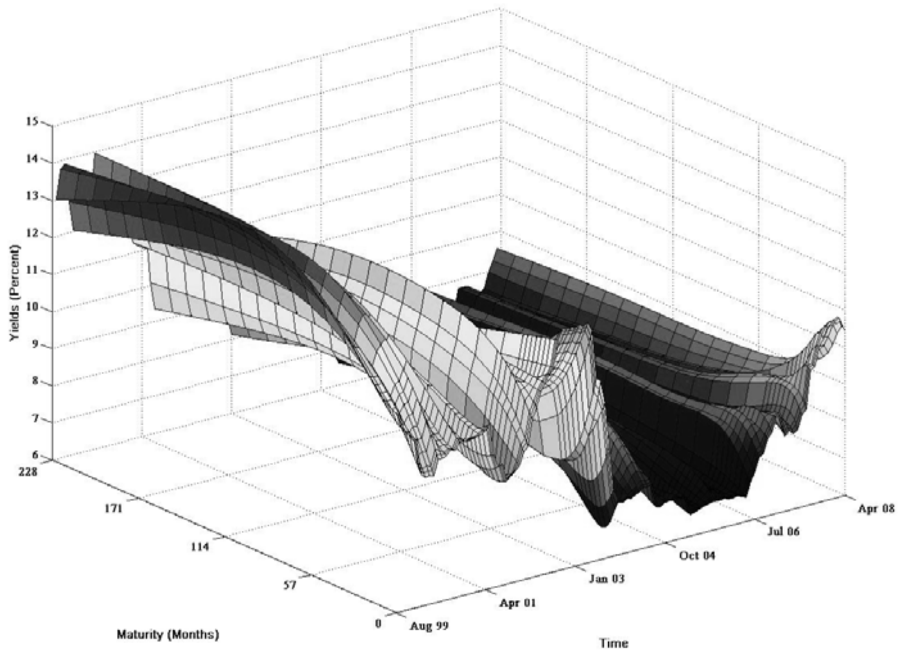

Figure 1. Yields curves, August 1999 to April 2008

2.3 THREE FACTOR MODEL ESTIMATION

We use the Perfect Fit Bond Curves, one of the five BEASSA Zero Coupon Yield Curve series of yield curves (see Bond exchange of South Africa, 2003a), with maturities 1, 2, 3, 6, 9, 12, 15, 18, 21, 24, 36, 48, 60, 72, 84, 96, 108, 120, 132, 144, 156, 168, 180, 192, 204, 216 and 228 months. The curves are derived from government bond data and the technical specifications are described in Bond exchange of South Africa (2003b). We use end-of-month data from August 1999 through to April 2008. Figure 1 provides a three dimensional plot of our yield curve data.

The variation in the level of the yield curve is visually apparent as is the variation in the slope and curvature of the yield curve. Descriptive statistics for the yields (mean, standard deviation, minimum, maximum and autocorrelations for one, twelfth and thirty months) are provided in Table 1. It is clear that the typical yield curve is humped shape with a positive hump at 120 months. The short rates are less volatile than the long rates but less persistent when comparing the twelfth month autocorrelation. This is the opposite compared to the U.S. term structure (See Diebold & Li, 2006). The level is persistent and varies moderately relative to its mean and the slope and the curvature are the least persistent. The slope is highly variable relative to its mean as is the curvature. In Figure 2 the median yield curve together with point-wise interquartile ranges are

Table 1. Descriptive statistics, yield curve

Maturity	Mean	Std. dev.	Minimum	Maximum	$\hat{\rho}(1)$	$\hat{\rho}(12)$	$\hat{\rho}(30)$
1	8.990	1.741	6.542	12.437	0.968	0.280	-0.231
2	8.988	1.713	6.554	12.383	0.969	0.277	-0.234
3	8.982	1.686	6.565	12.329	0.969	0.276	-0.233
6	8.957	1.605	6.604	12.172	0.967	0.288	-0.208
9	8.935	1.539	6.574	12.092	0.964	0.308	-0.161
12	8.930	1.493	6.531	12.058	0.959	0.328	-0.109
15	8.940	1.467	6.592	12.024	0.955	0.347	-0.057
18	8.960	1.455	6.708	11.989	0.952	0.365	-0.010
21	8.987	1.454	6.754	11.954	0.950	0.382	0.032
24	9.016	1.462	6.771	11.918	0.948	0.399	0.069
36	9.142	1.539	6.841	12.107	0.947	0.455	0.161
48	9.262	1.628	6.912	12.615	0.950	0.492	0.197
60	9.363	1.698	6.980	12.959	0.953	0.515	0.210
72	9.442	1.746	7.026	13.190	0.954	0.528	0.215
84	9.500	1.779	7.055	13.350	0.955	0.537	0.215
96	9.537	1.803	7.072	13.466	0.956	0.543	0.213
108	9.554	1.823	7.076	13.553	0.956	0.548	0.207
120	9.551	1.841	7.069	13.621	0.956	0.553	0.201
132	9.530	1.859	7.051	13.676	0.956	0.558	0.194
144	9.493	1.879	7.023	13.721	0.957	0.563	0.185
156	9.444	1.900	6.987	13.759	0.957	0.569	0.176
168	9.388	1.923	6.945	13.792	0.957	0.575	0.167
180	9.328	1.946	6.898	13.820	0.958	0.580	0.156
192	9.265	1.970	6.847	13.845	0.958	0.585	0.145
204	9.202	1.995	6.794	13.866	0.959	0.590	0.134
216	9.138	2.021	6.698	13.886	0.959	0.594	0.123
228	9.076	2.047	6.581	13.903	0.960	0.598	0.112
Level	9.203	1.626	6.818	12.821	0.962	0.606	0.170
Slope	-0.093	2.015	-3.760	4.079	0.962	0.176	-0.377
Curvature	-0.026	1.102	-2.453	2.847	0.868	-0.017	-0.047

displayed. The humped shaped pattern, with short rates less volatile than long rate, is apparent.

As in Diebold et al. (2006), the yields-only model forms a state-space system, with a VAR(1) transition equation summarising the dynamics of the vector of latent variables, and a linear measurement equation relating the observed yields to the state vector as described above. In the entire model there are 46 parameters that need to be estimated by

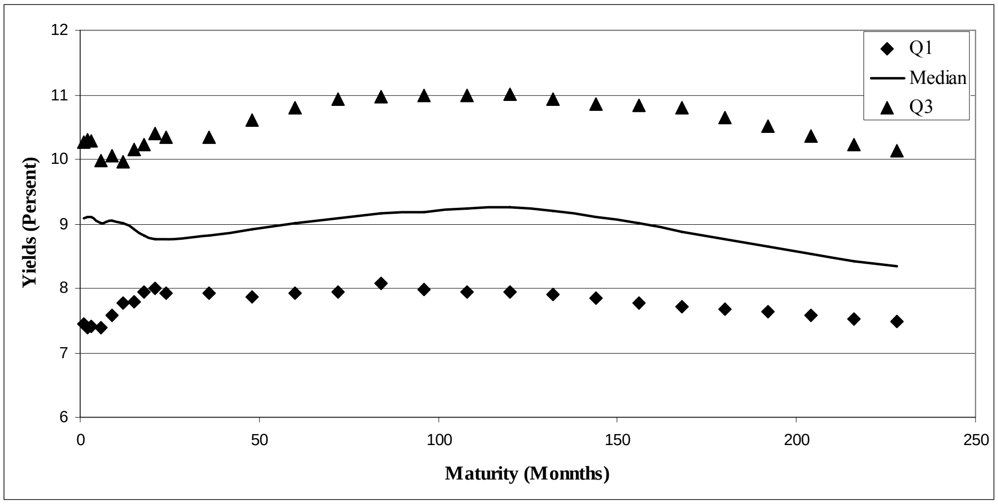

Figure 2. Median yield curve with point-wise interquartile range

the numerical optimisation of the relevant likelihood function. Let ψ be the vector of all parameters that need to be estimated. These parameters are the nine parameters contained in transition matrix A, the three parameters contained in the mean state vector μ, and the one parameter λ contained in the measurement matrix Λ.

Furthermore the transition disturbance covariance matrix Q contains six parameters, and the measurement disturbance covariance matrix H contains 27 parameters (one variance for each of the 27 yields). Given that the matrices A and Λ are affine and assuming that the distributions of η_t, ε_t and f_0 are normal, the model is referred to as a linear Gaussian state-space model (Lemke, 2006).

It follows by assumption that the transition density $p\left(f_{t+1} \mid f_t\right)$ and the measurement density $p\left(y_t \mid f_t\right)$ are jointly normal. This implies that the prediction and filtering densities are normal,

$$f_t \mid \Upsilon_{t-1} \sim N\left(\hat{f}_{t|t-1}, \Sigma_{t|t-1}\right),$$

$$f_t \mid \Upsilon_t \sim N\left(\hat{f}_{t|t}, \Sigma_{t|t}\right),$$

$$y_t \mid \Upsilon_{t-1} \sim N\left(\hat{y}_{t|t-1}, F_t\right),$$

where $\Upsilon_t = \left\{y_1, \ldots, y_t\right\}$ is taken to be the sequence of observations available for estimation and $\hat{f}_{t|t-1}$, $\hat{f}_t$, $\hat{y}_{t|t-1}$ and $\Sigma_{t|t-1}$, $\Sigma_{t|t}$, F_t the sequences of conditional means, and covariance matrices respectively. These quantities can be obtained by employing the Kalman filter for a given set of parameters ψ.

The Kalman filter algorithm can be described as follows (see Lemke, 2006):

Step 1: Set $\hat{f}_{0|0} = \bar{f}_0$, $\Sigma_{0|0} = \bar{\Sigma}_0$ and set $t = 0$.

Step 2: $\hat{f}_{t-1|t-1}$ and $\Sigma_{t-1|t-1}$ are given values, but y_t has not been observed yet. Compute

$$\left(\hat{f}_{t|t-1} - \mu\right) = A\left(\hat{f}_{t-1|t-1} - \mu\right),$$

$$\Sigma_{t|t-1} = A\Sigma_{t-1|t-1}A' + Q,$$

$$\hat{y}_{t|t-1} = \Lambda\hat{f}_{t|t-1} \text{ and}$$

$$F_t = \Lambda\Sigma_{t|t-1}\Lambda' + H.$$

Step 3: y_t has been observed. Compute

$$K_t = \Sigma_{t|t-1}\Lambda' F_t^{-1},$$

$$\hat{f}_{t|t} = \hat{f}_{t|t-1} + K_t\left(y_t - \hat{y}_{t|t-1}\right),$$

$$\Sigma_{t|t} = \Sigma_{t|t-1} - K_t\Lambda\Sigma_{t|t-1}.$$

Step 4: If $t < T$, set $t := t + 1$, and go to Step 2; else, stop.

Hence the Kalman filter delivers the sequence of means and covariance matrices for the conditional distributions of interest for a given set of parameters ψ. The Kalman filter is initialised by setting $\bar{f}_0$ and $\bar{\Sigma}_0$ to the unconditional mean and unconditional covariance matrix of the state vector respectively. Under the normality assumption, the distribution of y_t conditional on Υ_{t-1} is the N-dimensional normal distribution with mean $\hat{y}_{t|t-1}$ and covariance matrix F_t. The conditional density of y_t given Υ_{t-1} and ψ can be written as (see Lemke, 2006)

$$p\left(y_t \mid \Upsilon_{t-1};\psi\right) = \left[\left(2\pi\right)^{N/2}\sqrt{\left|F_t\right|}\right]^{-1} \cdot \exp\left[-\tfrac{1}{2}\left(y_t - \hat{y}_{t|t-1}\right)' F_t^{-1}\left(y_t - \hat{y}_{t|t-1}\right)\right].$$

Accordingly, the log-likelihood function becomes

$$\ln L\left(\psi\right) = -\frac{NT}{2}\log 2\pi - \frac{1}{2}\sum_{t=1}^{T}\log\left|F_t\right| - \frac{1}{2}\sum_{t=1}^{T} v_t' F_t^{-1} v_t,$$

where $v_t = \left(y_t - \hat{y}_{t|t-1}\right)$ is the vector of prediction errors.

For a given set of parameters ψ, the Kalman filter is used to compute the prediction errors v_t and their covariance matrix F_t, after which the log-likelihood function is computed. The parameters are estimated by maximising the log-likelihood function, using either the Nelder-Mead Simplex or Newton-Raphson algorithms.

Table 2. Three-factor yields-only model estimates (Bold entries denote parameters estimates significant at 5 percent, standard errors appear in parentheses)

	L_{t-1}	S_{t-1}	C_{t-1}	μ
L_t	**0.956** (0.026)	0.004 (0.022)	-0.010 (0.019)	**8.353** (1.154)
S_t	**0.085** (0.031)	**0.958** (0.027)	**0.129** (0.023)	0.383 (1.153)
C_t	**-0.154** (0.073)	-0.120 (0.063)	**0.856** (0.053)	-0.701 (0.758)

Table 3. Estimated Q matrix (Bold entries denote parameters estimates significant at 5 percent, standard errors appear in parentheses)

	L_t	S_t	C_t
L_t	**0.143** (0.020)	**-0.136** (0.022)	-0.046 (0.040)
S_t		**0.214** (0.030)	0.061 (0.049)
C_t			**1.136** (0.158)

For more details on Kalman filtering see Harvey (1989) and Lemke (2006). Non-negativity constraints are imposed on all the variances. As in Diebold et al. (2006), we obtain starting parameters using the two-step Diebold-Li method and initialising the variances to 1.0. As in Diebold & Li (2006) we initialise the value of λ at 0.0609 to maximise the loading on the curvature factor at exactly 30 months, i.e. the maturity at which the hump occurs in the yield curve.

In Table 2 and Table 3 we present the estimation results for the three-factor yields-only model. In Table 2 the estimate of the A matrix indicates the highly persistent dynamics of L_t, S_t and C_t, with estimated own lag coefficients 0.956, 0.958 and 0.856 respectively. Cross factor dynamics between S_t and L_t, S_t and C_t, and C_t and L_t appear to be important with statistical significant effects. The estimates indicate that persistence decreases in L_t, S_t ,and C_t as measured by the diagonal elements. The mean of the level is approximately 8 percent, the mean of the slope and the mean of the curvature does not seem statistical significant different from zero and appear to be reasonable. The largest eigenvalue of the A matrix 0.96, this ensures the stationarity of the system. In Table 3 the estimates of the Q matrix indicate that transitional shock

Table 4. Summary of statistics for predicted errors of yields

Maturity	Yields-only				Yields-macro			
	3 Factor		4 Factor		3 Factor		4 Factor	
	Mean	Std	Mean	Std	Mean	Std	Mean	Std
1	-0.0265	0.3633	-0.0147	0.4762	0.00499	0.45207	-0.00356	0.41848
2	0.0002	0.3484	0.0023	0.4578	0.00817	0.44751	-0.01295	0.40813
3	0.0190	0.3460	0.0139	0.4544	0.00619	0.32912	0.03356	0.32738
6	0.0406	0.3596	0.0246	0.4736	-0.02549	0.35777	0.04737	0.33934
9	0.0372	0.3793	0.0209	0.5000	0.00047	0.34028	0.04164	0.33493
12	0.0298	0.3996	0.0188	0.5231	0.01876	0.33617	0.03652	0.33680
15	0.0248	0.4173	0.0212	0.5414	0.03973	0.34937	0.02497	0.35622
18	0.0205	0.4332	0.0240	0.5564	0.03634	0.36995	0.01856	0.38057
21	0.0171	0.4475	0.0264	0.5691	0.02913	0.38991	0.01649	0.40204
24	0.0138	0.4602	0.0267	0.5796	0.02419	0.40648	0.01736	0.41974
36	0.0066	0.4992	0.0115	0.6069	0.01982	0.42086	0.01931	0.43520
48	0.0144	0.5202	-0.0136	0.6194	0.01649	0.43349	0.02158	0.44886
60	0.0308	0.5285	-0.0352	0.6254	0.01308	0.44459	0.02343	0.46086
72	0.0472	0.5309	-0.0483	0.6288	0.00570	0.47664	0.02375	0.49547
84	0.0577	0.5323	-0.0521	0.6305	0.01297	0.49193	0.01631	0.51160
96	0.0585	0.5350	-0.0487	0.6310	0.02864	0.49666	0.00857	0.51633
108	0.0465	0.5398	-0.0417	0.6311	0.04441	0.49728	0.00546	0.51626
120	0.0202	0.5469	-0.0346	0.6317	0.05450	0.49787	0.00824	0.51527
132	-0.0202	0.5571	-0.0296	0.6330	0.05513	0.50052	0.01532	0.51502
144	-0.0733	0.5716	-0.0276	0.6349	0.04336	0.50604	0.02373	0.51603
156	-0.1358	0.5910	-0.0277	0.6367	0.01775	0.51479	0.03088	0.51807
168	-0.2038	0.6157	-0.0277	0.6383	-0.02155	0.52752	0.03526	0.52059
180	-0.2746	0.6454	-0.0268	0.6396	-0.07321	0.54507	0.03660	0.52321
192	-0.3467	0.6794	-0.0248	0.6408	-0.13401	0.56809	0.03615	0.52592
204	-0.4189	0.7169	-0.0217	0.6419	-0.20009	0.59659	0.03541	0.52861
216	-0.4900	0.7572	-0.0173	0.6431	-0.26898	0.62999	0.03514	0.53130
228	-0.5591	0.7997	-0.0112	0.6450	-0.33908	0.66765	0.03560	0.53413

volatility increases as we move from L_t to S_t to C_t as measured by the diagonal elements. There is one significant covariance term between L_t and S_t in the Q matrix. The estimate for λ is 0.089 which implies that the loading on the curvature factor is maximised at a maturity of 20.15 months. This can be seen in Figure 2, where the first (inverted) hump occurs at around the maturity of 20 months.

Table 4 contains the means and standard deviations of the predicted errors. The three-factor yields-only model fits the yield curve reasonably well in the short maturities

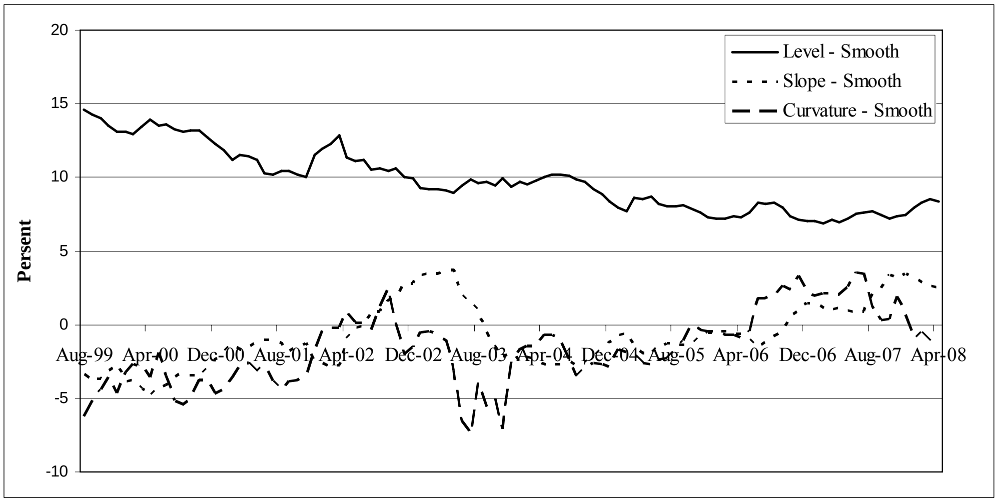

Figure 3. Estimates of the level, slope and curvature factors

but less in the longer maturities, with the standard deviation also increasing for longer maturities. This is similar to the yields-only model of Diebold et al. (2006).

We use the Kalman filter fixed-interval smoothing algorithm to obtain optimal extractions of the latent level, slope and curvature factors. The algorithm consists of a set of recursions which start with the final quantities given by the Kalman filter and work backwards (Harvey, 1989). The equations are

$$\hat{f}_{t|T} = \hat{f}_{t|t} + \Sigma_t^* \left(\hat{f}_{t+1|T} - \hat{f}_{t+1|t} \right) \text{ and}$$

$$\Sigma_{t|T} = \Sigma_{t|t} + \Sigma_t^* \left(\Sigma_{t+1|T} - \Sigma_{t+1|t} \right) \Sigma_t^{*\prime},$$

where $\Sigma_t^* = \Sigma_{t|t} A' \Sigma_{t+1|t}^{-1}$. Figure 3 plots the three estimated factors together and in Figure 4 to Figure 6 the three factors together with various empirical proxies and related macro-economic variables. The level factor in Figure 4 is positive in the neighbourhood of 8 percent and displays persistence. The slope and the curvature factors vary around zero with positive and negative values and appear less persistent. The slope factor is more persistent than the curvature factor but has a lower variance. The correlation between the slope and curvature factors is 0.47.

Figure 4 displays the estimated level factor and two linked comparison series. The first one is a commonly used empirical proxy for the level factor namely the average of the short-, medium- and long-term yields, $\left(y(3) + y(24) + y(228)\right)/3$. The second is the annual percentage change in the inflation index. There is a high correlation of 0.84 between the level factor and the empirical proxy. The correlation between the level factor and the inflation is 0.54. According to Diebold et al. (2006) this is consistent with the Fisher equation, which suggests a link between the yield curve and inflation.

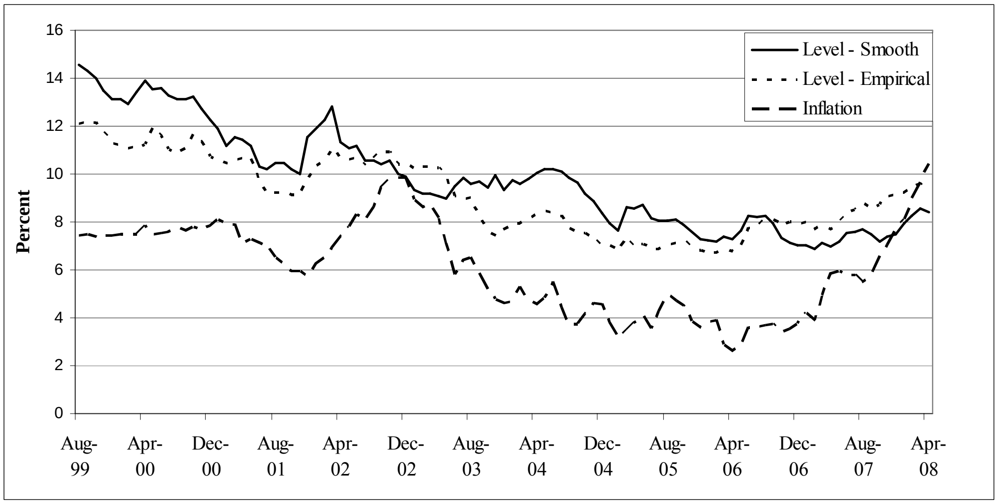

Figure 4. Three-factor yields-only model level factor and empirical estimates

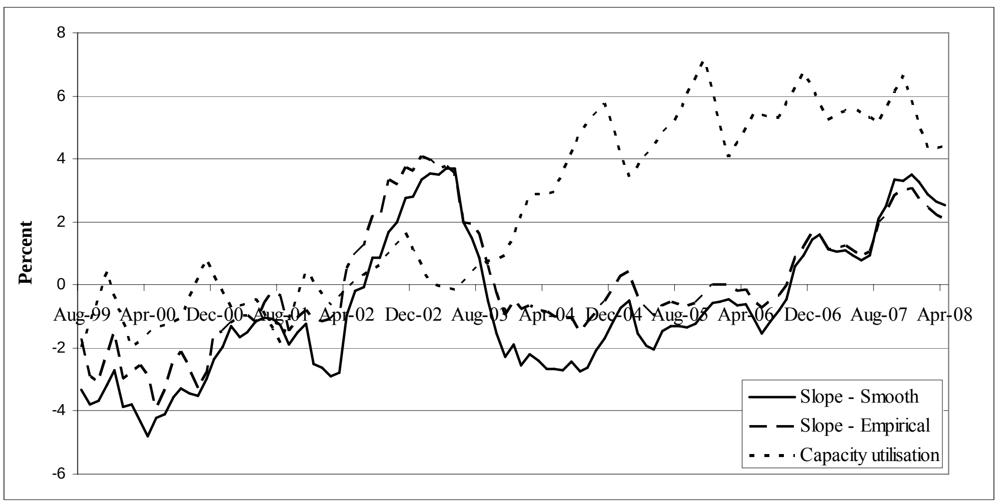

Figure 5. Three-factor yields-only model slope factor and empirical estimates

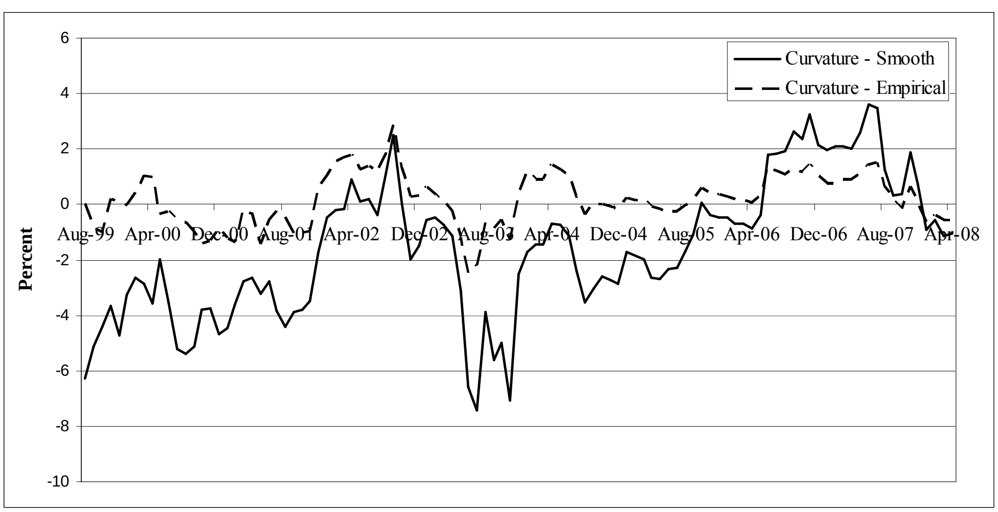

Figure 6. Three-factor yields-only model curvature factor and empirical estimates

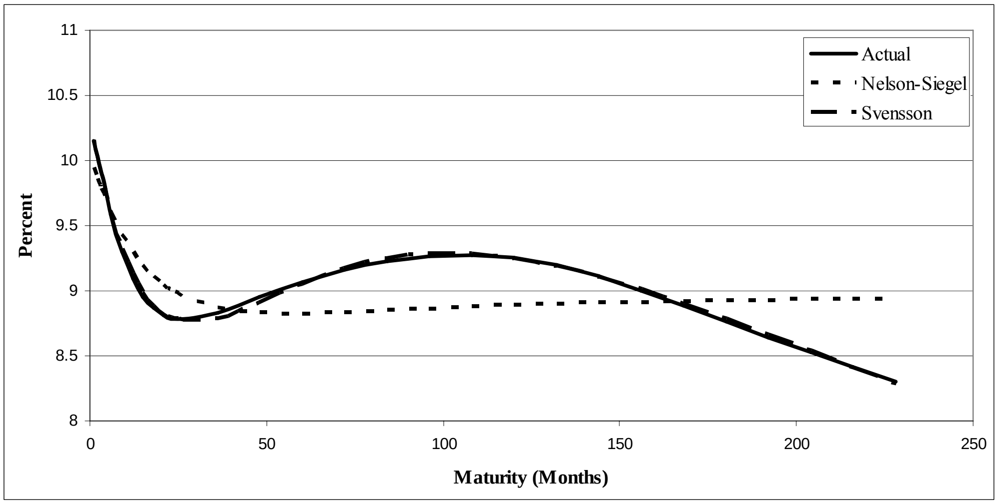

Figure 7. Nelson-Siegel fit versus Svensson fit of the yield curve

Figure 5 also displays the estimated slope factor and two linked comparison series. The first is the empirical proxy for the slope factor namely the difference between the short- and long-term yields, $y(3)-y(228)$. The second is an indicator of macro-economic activity namely the demeaned manufacturing capacity utilisation. There is a high correlation of 0.97 between the slope factor and the empirical proxy. The correlation between the slope factor and the capacity utilisation is 0.31. Diebold et al. (2006) suggest that, as with the level factor, there is a connection between the yield curve and the cyclical dynamics of the economy.

Furthermore, Figure 6 displays the curvature factor and the empirical proxy for the curvature of the yield curve, which is $2y(24)-y(3)-y(228)$. There is a correlation of 0.79 between the curvature factor and the empirical proxy. Diebold et al. (2006) report no reliable macro-economic links to the curvature factor.

2.4 FOUR-FACTOR MODEL ESTIMATION

In Table 4 we have shown that the three-factor model fits the yield curve reasonably well in the short maturities but less in the longer maturities. We extend the three-factor model to a four-factor model using the Svensson representation (Svensson, 1994) of the yield curve

$$y(\tau)=\beta_1+\beta_2\left(\frac{1-e^{-\lambda_1\tau}}{\lambda_1\tau}\right)+\beta_3\left(\frac{1-e^{-\lambda_1\tau}}{\lambda_1\tau}-e^{-\lambda_1\tau}\right)+\beta_4\left(\frac{1-e^{-\lambda_2\tau}}{\lambda_2\tau}-e^{-\lambda_2\tau}\right).$$

where $y(\tau)$ is the zero coupon yield with maturity τ and β_1, β_2, β_3, β_4, λ_1 and λ_2 are model parameters. Figure 7 illustrates an example fit of both the Nelson-Siegel curve the Svensson curve on an arbitrary yield curve in our dataset. It is clearly visible that the

Svensson curve is more flexible and provides a better fit to the South African term structure than the Nelson-Siegel curve.

We rewrite Svensson the representation as

$$y_t(\tau) = L_t + S_t\left(\frac{1-e^{-\lambda_1\tau}}{\lambda_1\tau}\right) + C_t^1\left(\frac{1-e^{-\lambda_1\tau}}{\lambda_1\tau} - e^{-\lambda_1\tau}\right) + C_t^2\left(\frac{1-e^{-\lambda_2\tau}}{\lambda_2\tau} - e^{-\lambda_2\tau}\right)$$

where L_t, S_t, C_t^1 and C_t^2 are the, time-varying unobserved factors, β_1, β_2, β_3 and β_4, respectively. We interpret the factors L_t, S_t, C_t^1 and C_t^2 as level, slope, curvature 1 and curvature 2. The state-space system can be rewritten as

$$(f_t - \mu) = A(f_t - \mu) + \eta_t,$$

$$y_t = \Lambda f_t + \varepsilon_t,$$

$$\begin{pmatrix}\eta_t\\ \varepsilon_t\end{pmatrix} \sim WN\left[\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}Q & 0\\ 0 & H\end{pmatrix}\right],$$

where $f_t = \left(L_t, S_t, C_t^1, C_t^2\right)'$. The dimensions of A, μ, η_t and Q are increased as appropriate. Λ is changed to be

$$\Lambda = \begin{pmatrix} 1 & \frac{1-e^{-\tau_1\lambda_1}}{\tau_1\lambda_1} & \frac{1-e^{-\tau_1\lambda_1}}{\tau_1\lambda_1} - e^{-\tau_1\lambda_1} & \frac{1-e^{-\tau_1\lambda_2}}{\tau_1\lambda_2} - e^{-\tau_1\lambda_2} \\ 1 & \frac{1-e^{-\tau_2\lambda_1}}{\tau_2\lambda_1} & \frac{1-e^{-\tau_2\lambda_1}}{\tau_2\lambda_1} - e^{-\tau_2\lambda_1} & \frac{1-e^{-\tau_2\lambda_2}}{\tau_2\lambda_2} - e^{-\tau_2\lambda_2} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \frac{1-e^{-\tau_N\lambda_1}}{\tau_N\lambda_1} & \frac{1-e^{-\tau_N\lambda_1}}{\tau_N\lambda_1} - e^{-\tau_N\lambda_1} & \frac{1-e^{-\tau_N\lambda_2}}{\tau_N\lambda_2} - e^{-\tau_N\lambda_2} \end{pmatrix}.$$

In Table 5 and Table 6 we present the estimation results for the four-factor yields-only model. In Table 5 the estimate of the A matrix indicates high persistent own dynamics of L_t, S_t, C_t^1 and C_t^2, with estimated own lag coefficients of 0.927, 1.024, 0.702 and 1.103 respectively. Some cross factor dynamics seem significantly important. The estimates indicate the persistence in L_t and C_t^1 decreases and an increases in S_t and C_t^2, as measured by the diagonal elements. The mean of the level is approximately 6.4 percent and is statistical significant different from zero. The mean of the slope is 2.357 percent, the mean of the first curvature factor is -0.239 percent and the mean of the second curvature factor 6.293 percent, which are not statistical significant different from zero. The largest eigenvalue of the A matrix 0.95, this ensures the stationarity of the

Table 5. Four-factor yields-only model estimates (Bold entries denote parameters estimates significant at 5 percent, standard errors appear in parentheses)

	L_{t-1}	S_{t-1}	C^1_{t-1}	C^2_{t-1}	μ
L_t	**0.927** (0.050)	0.031 (0.039)	-0.049 (0.040)	0.006 (0.014)	**6.429** (1.054)
S_t	0.080 (0.049)	**1.024** (0.045)	**0.112** (0.039)	-0.016 (0.017)	2.357 (1.376)
C^1_t	**-0.251** (0.017)	**-0.193** (0.009)	**0.702** (0.053)	**0.071** (0.023)	-0.239 (0.391)
C^2_t	**0.231** (0.061)	-0.047 (0.118)	-0.134 (0.097)	**1.103** (0.030)	6.293 (4.028)

Table 6. Four-factor yields-only estimated Q matrix (Bold entries denote parameters estimates significant at 5 percent, standard errors appear in parentheses)

	L_t	S_t	C^1_t	C^2_t
L_t	**0.473** (0.031)	0.007 (0.025)	-0.006 (0.117)	-0.048 (0.135)
S_t		**0.630** (0.032)	0.151 (0.133)	-0.045 (0.071)
C^1_t			**1.146** (0.152)	-0.022 (0.212)
C^2_t				**4.371** (0.678)

system. In Table 6 the estimates indicate an increase in the transitional shock volatility as we move from L_t, S_t, C^1_t and C^2_t. The estimate for λ_1 is 0.087 which implies that the loading on the first curvature factor is maximised at a maturity of 20.61 months and the estimate for λ_2 is 0.015 which implies that the loading on the second curvature factor is maximised at a maturity of 119.55 months. Again in Figure 2 it can be seen that the first hump is at about 20 months and the second hump at 120 months.

As shown in Table 4, the four-factor yields-only model improves on the means of the measurement errors, especially for the long maturities. Again we plot the estimated level and slope factors against empirical proxies and macro-economic factors in Figures 8 and Figure 9. In Figure 8 we plot the estimated level factor against the empirical proxy and annual percentage change in the inflation index. There is a correlation of 0.68 between the estimated level factor and the empirical proxy. The correlation between the estimated level and the inflation is 0.39, which again suggests that inflation is linked to dynamics of the yield curve. Figure 9 shows the estimated slope curve together with the

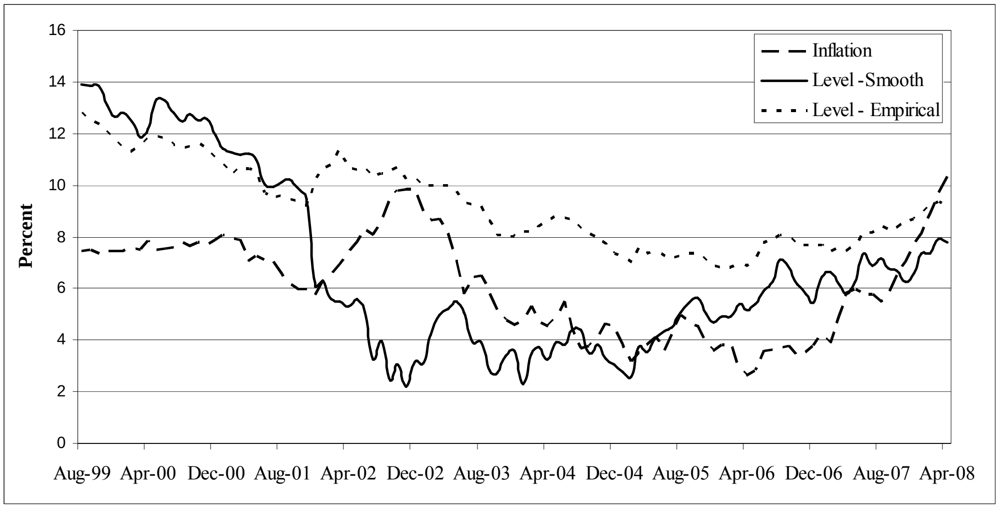

Figure 8. Four-factor yields-only model level factor and empirical estimates

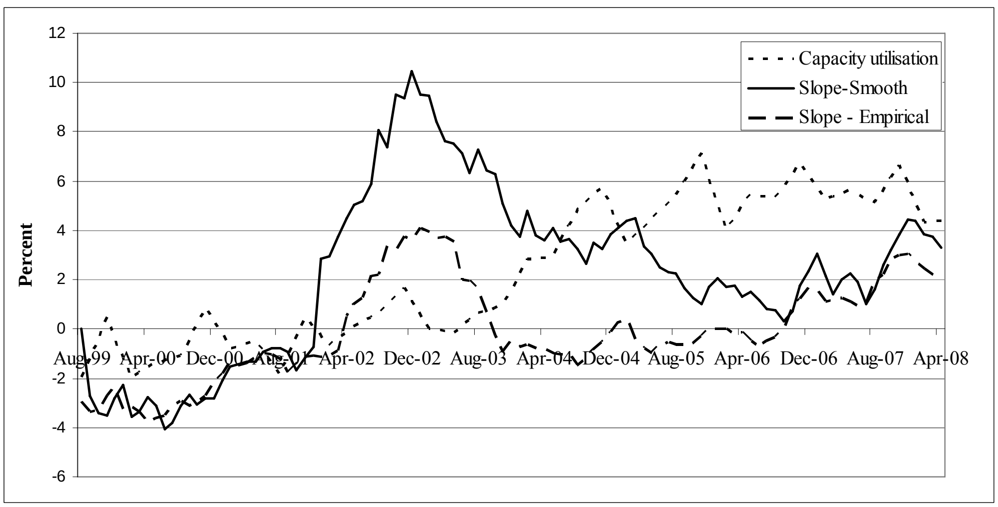

Figure 9. Four-factor yields-only model slope factor and empirical estimates

empirical proxy and demeaned manufacturing capacity utilisation. There is a 0.85 correlation between the estimated slope factor and the empirical proxy, and a 0.23 correlation between the estimated slope factor and the capacity utilisation. This also suggests a link between the capacity utilisation and the dynamic of the yield curve.

3. MACRO-ECONOMIC MODEL

In this section we relate the four unobserved factors, level, slope and the two curvature factors, that provide a good representation of the yield curve, to the macro-economic variables. This can be done by extending the state-space model above. We also present out-of-sample forecasting results.

3.1 YIELDS-MACRO MODEL

We include the following three macro economic variables: manufacturing capacity utilisation (CU_t), which represents the level of real economic activity relative to potential; the annual percentage change in the inflation index (IF_t), which represent the inflation rate; and the repo-rate (RR_t), which represents the monetary policy instrument. According to Diebold et al. (2006) these three macro-economic variables are considered to be the minimum set of fundamentals needed to capture the basic macro-economic dynamics (see also Rudebusch & Svensson, 1999 and Kozicki & Tinsley, 2001).

We extend the four-factor yields-only model to incorporate the three macro-economic variables by adding the variables to the set of state variables. The state-space system is rewritten as

$$\begin{pmatrix} (f_t-\mu) \\ (x_t-\nu) \end{pmatrix} = \begin{pmatrix} A_{11}(f_{t-1}-\mu) \\ A_{21}(f_{t-1}-\mu) \end{pmatrix} + \begin{pmatrix} A_{12}(x_{t-1}-\nu) \\ A_{22}(x_{t-1}-\nu) \end{pmatrix} + \begin{pmatrix} \eta_t \\ \gamma_t \end{pmatrix},$$

$$y_t = \Lambda f_t + \varepsilon_t,$$

$$\begin{pmatrix} \eta_t \\ \gamma_t \\ \varepsilon_t \end{pmatrix} \sim WN\left[\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} Q & K & 0 \\ K' & J & 0 \\ 0 & 0 & H \end{pmatrix}\right],$$

where $f_t = \left(L_t, S_t, C_t^1, C_t^2\right)'$ and $x_t = \left(CU_t, IF_t, RR_t\right)'$. Where A_{11}, A_{12}, A_{21}, A_{22}, μ, ν, η_t, γ_t, Q, K and J have appropriate dimensions. Λ stays unchanged. This is consistent with the view that only four factors are needed to distil the information in the yield curve (Diebold et al., 2006). In the four-factor yields-macro model the matrix

$$\begin{pmatrix} Q & K \\ K' & J \end{pmatrix}$$

is assumed to be non-diagonal and H is assumed to be diagonal.

As previously stated it follows by assumption that the transition density $p(f_{t+1} \mid f_t)$ and $p(x_{t+1} \mid x_t)$ and the measurement density $p(y_t \mid f_t)$ are jointly normal. This implies that the prediction and filtering densities are normal,

$$\begin{pmatrix} f_t \\ x_t \\ y_t \end{pmatrix} \Bigg| G_{t-1} \sim N\left(\begin{pmatrix} \hat{f}_{t|t-1} \\ \hat{x}_{t|t-1} \\ \hat{y}_{t|t-1} \end{pmatrix}, \begin{pmatrix} \Sigma_{t|t-1}^{ff} & \Sigma_{t|t-1}^{fx} & \Sigma_{t|t-1}^{fy} \\ \Sigma_{t|t-1}^{xf} & \Sigma_{t|t-1}^{xx} & \Sigma_{t|t-1}^{xy} \\ \Sigma_{t|t-1}^{yf} & \Sigma_{t|t-1}^{yx} & \Sigma_{t|t-1}^{yy} \end{pmatrix}\right),$$

$$f_t \mid G_t \sim N\left(\hat{f}_{t|t}, \Sigma_{t|t}^{ff}\right),$$

where $G_t = \{y_1,\dots,y_t,x_1,\dots x_t\}$ is taken to be the sequence of observations available for estimation. These quantities can be obtained by employing the Kalman filter for a given set of parameters ψ.

The Kalman filter algorithm is updated as follows:

Step 1: Set $\hat{f}_{0|0} = \bar{f}_0$, $\Sigma^{ff}_{0|0} = \bar{\Sigma}_0$ and set $t = 0$.

Step 2: $\hat{f}_{t-1|t-1}$ and $\Sigma^{ff}_{t-1|t-1}$ are given values, but y_t and x_t has not been observed yet. Compute

$$\left(\hat{f}_{t|t-1} - \mu\right) = A_{11}\left(\hat{f}_{t-1|t-1} - \mu\right) + A_{12}\left(x_{t-1} - \nu\right),$$

$$\left(\hat{x}_{t|t-1} - \mu\right) = A_{11}\left(\hat{f}_{t-1|t-1} - \mu\right) + A_{12}\left(x_{t-1} - \nu\right),$$

$$\hat{y}_{t|t-1} = \Lambda \hat{f}_{t|t-1}$$

$$\Sigma^{ff}_{t|t-1} = A_{11}\Sigma^{ff}_{t-1|t-1}A'_{11} + Q,$$

$$\Sigma^{xx}_{t|t-1} = A_{21}\Sigma^{ff}_{t-1|t-1}A'_{21} + J,$$

$$\Sigma^{yy}_{t|t-1} = \Lambda\Sigma^{ff}_{t|t-1}\Lambda' + H,$$

$$\Sigma^{fx}_{t|t-1} = A_{11}\Sigma^{ff}_{t-1|t-1}A'_{21} + K,$$

$$\Sigma^{fy}_{t|t-1} = \Sigma^{ff}_{t|t-1}\Lambda' \text{ and}$$

$$\Sigma^{xy}_{t|t-1} = \Sigma^{fx}_{t|t-1}\Lambda'.$$

Step 3: y_t and x_t has been observed. Compute

$$\hat{f}_{t|t} = \hat{f}_{t|t-1} + \begin{pmatrix}\Sigma^{fx}_{t|t-1} & \Sigma^{fy}_{t|t-1}\end{pmatrix}\begin{pmatrix}\Sigma^{xx}_{t|t-1} & \Sigma^{xy}_{t|t-1}\\ \Sigma^{yx}_{t|t-1} & \Sigma^{yy}_{t|t-1}\end{pmatrix}^{-1}\begin{pmatrix}y_t - \hat{y}_{t|t-1}\\ x_t - \hat{x}_{t|t-1}\end{pmatrix},$$

$$\Sigma^{ff}_{t|t} = \Sigma^{ff}_{t|t-1} - \begin{pmatrix}\Sigma^{fx}_{t|t-1} & \Sigma^{fy}_{t|t-1}\end{pmatrix}\begin{pmatrix}\Sigma^{xx}_{t|t-1} & \Sigma^{xy}_{t|t-1}\\ \Sigma^{yx}_{t|t-1} & \Sigma^{yy}_{t|t-1}\end{pmatrix}^{-1}\begin{pmatrix}\Sigma^{fx}_{t|t-1}\\ \Sigma^{fy}_{t|t-1}\end{pmatrix}.$$

Step 4: If $t < T$, set $t := t + 1$, and go to Step 2; else, stop.

Accordingly, the log-likelihood function becomes

$$\ln L(\psi) = -\frac{NT}{2}\log 2\pi - \frac{1}{2}\sum_{t=1}^{T}\log\left|\begin{pmatrix}\Sigma^{xx}_{t|t-1} & \Sigma^{xy}_{t|t-1}\\ \Sigma^{yx}_{t|t-1} & \Sigma^{yy}_{t|t-1}\end{pmatrix}\right| - \frac{1}{2}\sum_{t=1}^{T}v_t'\begin{pmatrix}\Sigma^{xx}_{t|t-1} & \Sigma^{xy}_{t|t-1}\\ \Sigma^{yx}_{t|t-1} & \Sigma^{yy}_{t|t-1}\end{pmatrix}^{-1}v_t,$$

where $v_t = \begin{pmatrix}y_t - \hat{y}_{t|t-1}\\ x_t - \hat{x}_{t|t-1}\end{pmatrix}$ is the vector of prediction errors.

Table 7. Four-factor yields-macro model estimates (Bold entries denote parameters estimates significant at 5 percent, standard errors appear in parentheses)

	L_{t-1}	S_{t-1}	C_{1t-1}	C_{2t-1}	CU_{t-1}	IF_{t-1}	RR_{t-1}	μ
L_t	**0.516** (0.036)	**-0.369** (0.034)	0.002 (0.033)	-0.020 (0.024)	0.044 (0.061)	0.109 (0.092)	**0.288** (0.097)	**6.427** (2.318)
S_t	**0.503** (0.114)	**1.303** (0.084)	**0.116** (0.040)	0.018 (0.028)	-0.046 (0.071)	-0.018 (0.109)	**-0.353** (0.060)	2.746 (2.535)
C_{1t}	**0.250** (0.079)	**0.217** (0.044)	**0.867** (0.066)	0.030 (0.048)	-0.045 (0.116)	-0.037 (0.161)	-0.303 (0.177)	0.785 (1.546)
C_{2t}	**1.622** (0.045)	**1.534** (0.102)	0.098 (0.082)	**0.907** (0.079)	**-0.533** (0.168)	-0.245 (0.303)	**-1.543** (0.298)	**7.747** (3.771)
CU_t	0.163 (0.120)	0.108 (0.091)	0.014 (0.023)	0.019 (0.016)	**0.938** (0.041)	-0.086 (0.062)	**-0.119** (0.048)	**83.127** (1.067)
IF_t	**0.416** (0.037)	**0.261** (0.045)	**0.057** (0.023)	**0.032** (0.014)	0.053 (0.038)	**0.953** (0.060)	**-0.206** (0.075)	**6.387** (1.427)
RR_t	**0.503** (0.062)	**0.324** (0.060)	**0.130** (0.019)	0.019 (0.012)	-0.029 (0.033)	**0.103** (0.050)	**0.534** (0.071)	**10.013** (1.091)

Table 8. Four-factor yields-macro estimated Q matrix (Bold entries denote parameters estimates significant at 5 percent, standard errors appear in parentheses)

	L_t	S_t	C_{1t}	C_{2t}	CU_t	IF_t	RR_t
L_t	**0.525** 0.005	-0.001 0.036	0.000 0.087	0.000 0.112	0.000 0.077	0.000 0.019	0.000 0.042
S_t		**0.633** 0.096	0.000 0.060	0.000 0.171	0.000 0.036	0.000 0.051	0.000 0.011
C_{1t}			**1.512** 0.148	0.000 0.167	0.000 0.097	0.000 0.198	0.000 0.150
C_{2t}				**4.932** 1.249	0.000 0.100	0.000 0.195	0.000 0.165
CU_t					**0.210** 0.031	0.000 0.021	-0.001 0.011
IF_t						**0.199** 0.036	0.001 0.010
RR_t							**0.101** 0.013

In Table 7 and Table 8 we present the estimation results for the four-factor yields-macro model. The estimate of the A matrix again indicates high persistent own dynamics for S_t, C_t^1, C_t^2, CU_t and IF_t. Some of the cross factor dynamics are significantly important in most factors. The estimates also indicate an increase in the transitional shock volatility as we move from L_t, S_t, C_t^1 and C_t^2 all being statistical significant different from zero, and a decrease in the transitional shock volatility as we move from CU_t, IF_t and RR_t, all being statistical significant different from zero. There is a small change in the mean of the level, slope two curvature factors and appear to be reasonable. The largest eigenvalue of the A matrix 0.96, this ensures the stationarity of the system. None of the covariance terms in the Q matrix are significant.

As shown in Table 4, the four-factor yields-macro model improves on the means and standard deviations of the measurement errors. We also provide the means and standard deviations of the measurement errors for the three-factor yields-macro model, again the four-factor yields-macro model estimates the yield curve better than the three-factor yields-macro model. The estimates for the level, slope and two curvature factors of the four-factor yields-macro model are very similar to those of the four-factor yields-only model.

3.2 OUT-OF-SAMPLE TESTING

For scenario generation it is not only important to capture the dynamics of the yield curve well in-sample but it is also important to forecast the dynamics of the yield curve well out-of-sample. For this reason we estimate the four-factor yields-macro model on truncated date sets. Using the estimated parameters we forecast the yield curve recursively for one, two, three and four years ahead over the period of April 2003 through to April 2008, using monthly intervals. Diebold & Li (2006) model and forecast the Nielson-Siegel factors as univariate AR(1) processes for one, six and twelve month and outperforms other methods on all maturities. Thus we model and forecast the Svensson factors as univariate AR(1) processes in order to compare their model against our four-factor yields-macro model.

In Table 9 to Table 12 we present the out-of-sample forecasting results for maturities 3, 12, 60, 120 and 288 months. We define the forecast errors at time $t+h$ to be $y_{t+h}(\tau)-\hat{y}_{t+h}(\tau)$. We report the mean and standard deviation of the forecast errors. The four-factor yields-macro model outperforms the AR(1) model. The standard deviations for the AR(1) model are also larger than that of the four-factor yields-macro model. In particular the four year ahead forecast of the four-factor yields-macro model is better than that for the AR(1) model.

Table 9. One year out-of -sample forecasting results

Maturity	Four-Factor		Svensson - AR(1)		Four-Factor with repo-rate	
	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.
3	-1.053	1.562	-1.338	2.213	-0.395	1.164
12	-0.686	1.103	-1.281	1.909	-0.021	0.742
36	-0.706	0.605	-1.666	1.417	-0.046	0.408
60	-0.929	0.593	-2.036	1.166	-0.276	0.573
120	-1.009	0.741	-2.277	0.972	-0.364	0.837
180	-0.932	0.649	-2.292	0.969	-0.291	0.752
228	-0.854	0.524	-2.274	1.022	-0.217	0.621

Table 10. Two year out-of -sample forecasting results

Maturity	Four-Factor		Svensson - AR(1)		Four-Factor with repo-rate	
	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.
3	-0.881	2.004	-1.585	2.273	-0.178	1.564
12	-0.578	1.547	-1.576	1.951	0.164	1.235
36	-0.822	1.089	-2.124	1.763	-0.074	0.815
60	-1.195	1.000	-2.611	1.853	-0.459	0.638
120	-1.406	0.984	-2.947	1.951	-0.688	0.639
180	-1.289	0.954	-2.895	1.935	-0.592	0.622
228	-1.141	0.950	-2.784	1.900	-0.459	0.646

Table 11. Three year out-of -sample forecasting results

Maturity	Four-Factor		Svensson - AR(1)		Four-Factor with repo-rate	
	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.
3	-0.472	1.892	-1.235	2.013	0.120	1.539
12	-0.195	1.343	-1.290	1.720	0.417	1.134
36	-0.645	1.144	-2.061	2.233	-0.056	0.668
60	-1.173	1.460	-2.703	2.835	-0.605	0.827
120	-1.506	1.710	-3.166	3.284	-0.967	1.095
180	-1.394	1.638	-3.107	3.234	-0.882	1.040
228	-1.218	1.515	-2.954	3.088	-0.725	0.936

Table 12. Four year out-of -sample forecasting results

Maturity	Four-Factor		Svensson - AR(1)		Four-Factor with repo-rate	
	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.
3	0.067	1.258	-0.511	1.376	0.451	1.381
12	0.067	0.817	-0.887	1.682	0.469	1.069
36	-0.617	1.247	-1.907	3.382	-0.230	0.630
60	-1.212	2.220	-2.620	4.627	-0.842	1.555
120	-1.671	3.011	-3.212	5.663	-1.323	2.379
180	-1.573	2.860	-3.152	5.564	-1.243	2.260
228	-1.420	2.611	-3.004	5.308	-1.104	2.033

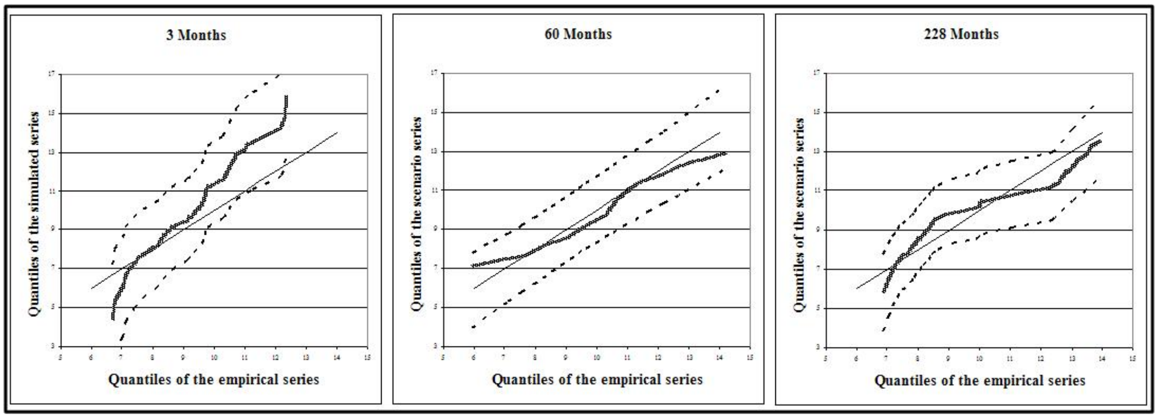

Figure 10. Quantile-quantile plots for maturities 3, 60 and 228 months

Also in Table 9 to Table 12 we present out-of-sample forecasting results where the repo-rate was included in the forecasting. As can be seen the forecasting error improves, especially in the long rates. The reasoning behind the inclusion of the repo-rate as input for the forecasting is that practitioners and long term investors usually have reasonably good forecasts or views on the future movements of the monetary policy instrument. By including this view a better forecast can be made.

In Figure 10 we present the quantile-quantile plots for maturities 3, 60 and 228 months. We set the quantiles of the empirical distribution against the quantiles obtained by averaging over a set of scenarios generated by the four-factor yields-macro model. The four-factor yields-macro model better reproduce the empirical distribution in the long rates than in the short rates.

4. SCENARIO GENERATION

In the next section we describe the scenario generation algorithm that we use to generate yield curve scenario trees for fixed income portfolio optimisation problems. We use the four-factor yields-macro model to generate yield curve scenarios. We discuss the existence of arbitrage in the scenario trees and propose a method to reduce arbitrage opportunities. We also demonstrate that the scenarios are stable by using back-testing.

4.1 YIELD CURVE SCENARIO GENERATION

We describe a procedure based on the parallel simulation and randomised clustering approach of Gülpinar et al. (2004) to generate a scenario tree which is the input for financial optimisation problems. The basic data structure is the scenario tree node, which contains a cluster of yield curve scenarios, one of which is designated as the centroid. The final tree consists of the centroids of each node, and their branch probabilities. Gülpinar et al. (2004) introduced a randomised clustering algorithm. This differs from the approach proposed by Dupacova et al. (2000) which determines clusters that are optimal by some measure. Our approach is to group the scenarios into equal groups rather than using a clustering approach as these approaches may need a very large number of scenarios to be generated at the root node to ensure sufficient scenarios at the leave nodes.

The specific scenario tree structure that we are interested in is a yield curve scenario tree. A T-period scenario tree structure is represented as a *tree-string* which is a string of integers specifying for each stage $s = 1, 2, \ldots, T$ the number of branches (or branching factor) for each node in that stage (see Dempster et al., 2006). This gives rise to balanced scenario trees, in which each sub tree in the same period has the same number of branches. Let k_s denote the branching factor for stage *s*, then Figure 11 gives an example of a scenario tree with a (3,2) tree-string, i.e. $k_1 = 3$ and $k_2 = 2$. Figure 12 illustrates the methods of scenario simulation, namely parallel and sequential. We use the parallel method for simulation as this method will produce more realistic extreme events in the scenario tree (Gülpinar et al., 2004).

The main steps of our algorithm can be outlined as follow:

Step 1: At $s = 0$ create a root node group containing N scenarios. Generate all the scenarios using Monte Carlo simulation and the four-factor yields-macro model. Each scenario is equally likely and consists of T sequential yield curves (in total $T \times N$ yield curves are generated).

Step 2: Set $s := s + 1$ and for each group in the previous stage, calculate the mean scenario and calculate the *relative position* (defined below) of each scenario with respect to the average scenario.

Step 3: For each group, sort the scenarios in descending distance order and group them into k_s equal sized groups.

Step 4: For each new group, find the scenario closest (in absolute value) to its centre, and designate it as the centroid. Assign a probability of $\left(\prod_{i=1}^{s-1} k_i\right)^{-1}$ to each centroid.

Step 5: If $s < T$, go to Step 2, else stop.

As a measure of *relative position* we calculate the "distance" between the discounting factors of the yield curve and that of the average by:

$$D = \sum_{\tau} \left(\frac{1}{\left(1 + y(\tau)\right)^{\tau}} - \frac{1}{\left(1 + y^{M}(\tau)\right)^{\tau}} \right),$$

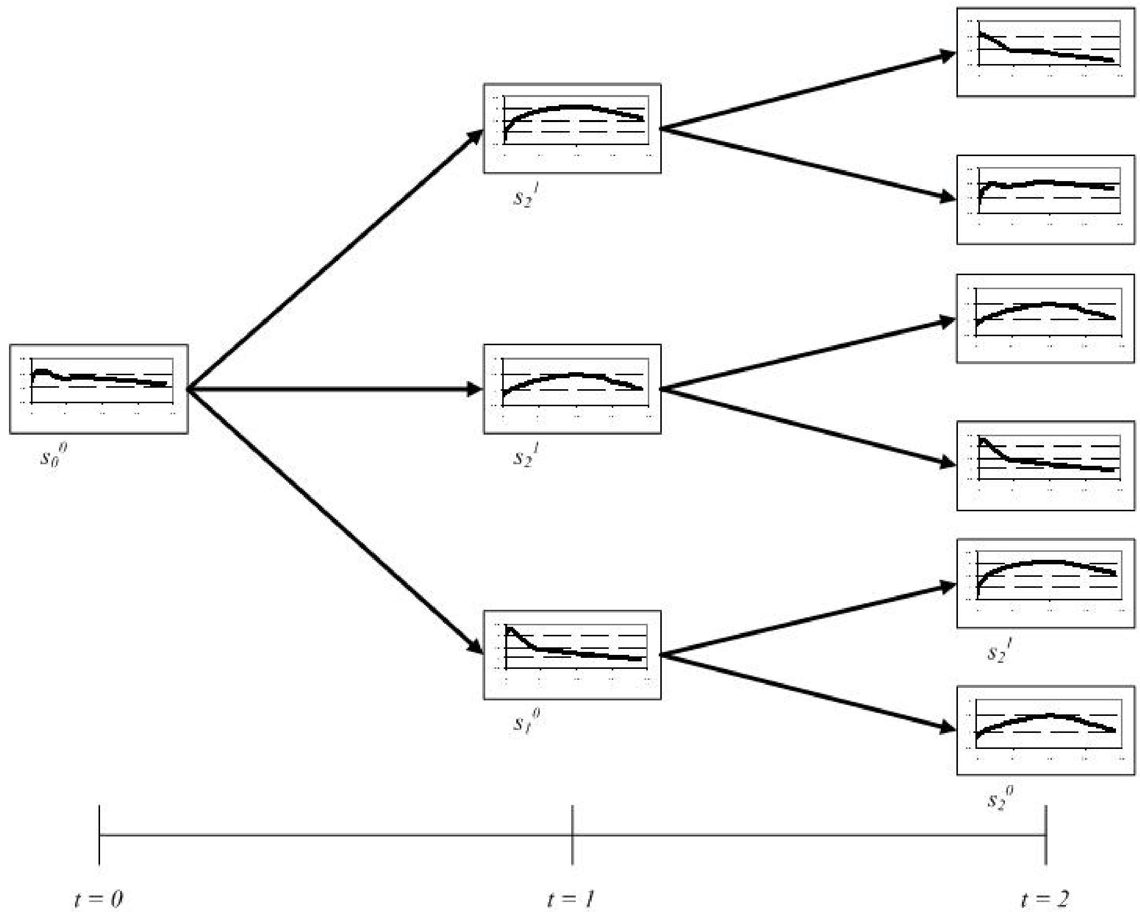

Figure 11. Graphic representation of scenarios

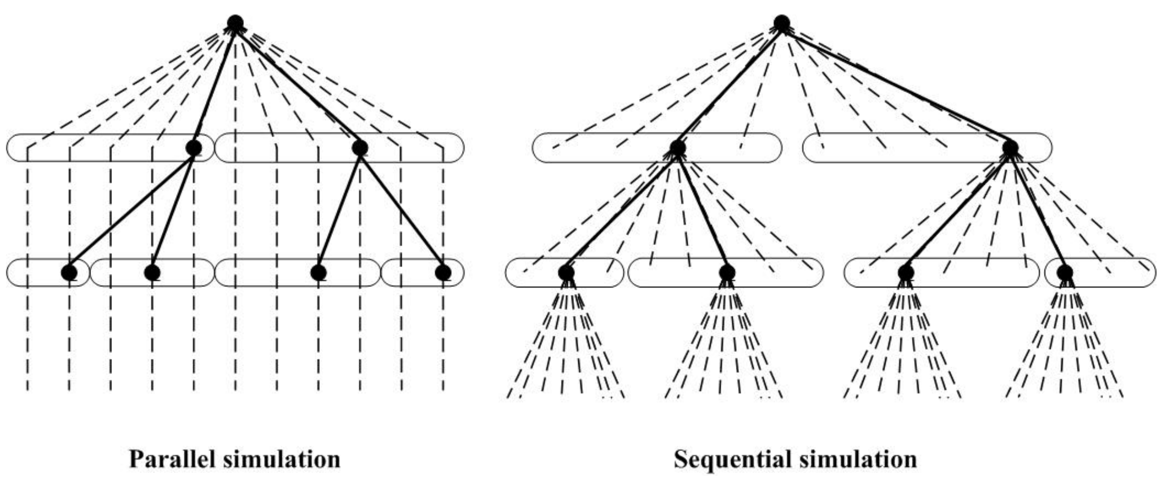

Figure 12. Two methods of simulating scenarios

where $y(\tau)$ is the zero rate with maturity τ and $y^M(\tau)$ the average zero rate with maturity τ. Note that the relative distance *D* can be negative and positive, which means that a yield curve can be positioned to the "left" or to the "right" of the average yield curve. Chueh (2002) discusses several other distance methods for interest rate sampling. Our relative distance method relates closely to the relative present value distance method in Chueh (2002). It is necessary to represent each group of scenarios with a single point, which becomes the data in the scenario tree. Gülpinar et al. (2004) argue that to prevent the scenario tree from containing scenarios that are not consistent with the simulation parameters, the centroid should not be taken to be the centre of the group, but rather the simulated scenario closest to the centre. We use the mean of the group as the notion of the centre, other notions of the centre that can be used is the median and the mode.

4.2 ARBITRAGE

Filipović (1999) and other researcher such as Diebold et al. (2005) showed that the Nelson-Siegel family of yield curve models does not impose absence of arbitrage, although these models estimates and forecasts the yield curve better than arbitrage-free models (Duffee, 2002 noted that the canonical affine arbitrage-free models demonstrate disappointing out-of-sample performance). In light of this, the scenarios generated are not arbitrage free. Klaassen (2002) shows that arbitrage opportunities can be detected *ex post* by checking for solutions to a set of linear constraints or be excluded by including non-linear constraints in the scenario generation process. Christensen et al. (2007) derives a class of arbitrage-free affine dynamic term structure models that approximate the Nelson-Siegel yield curve specification. Christensen et al. (2008) extends these models to include the Svensson extension of the Nelson-Siegel yield curves.

We propose a method to reduce the presence of arbitrage *ex post*, without extending our models to the class of arbitrage-free models. We reduce the presence of arbitrage *ex post* as to excluding it, by means of including non-linear constraints during the scenario generation process. This approach has no additional effect on the computational difficulty of the model estimation process and the data requirements. As the scenario generation process is a discrete approximation of the continuous evolution of the term structure, extending the models to a class of arbitrage-free models will not ensure the exclusion of arbitrage in the generated scenarios.

Klaassen (2002) proposes linear constraints for two types of arbitrage. Ingersoll (1987) distinguishes these two types of arbitrage. The first type is an opportunity to construct a zero-investment portfolio that has nonnegative payoffs in all states of the world, and a strictly positive payoff in at least one state. The second type is an opportunity to construct a negative investment portfolio (i.e. providing an immediate positive cash flow) that generates a nonnegative payoff in all future states of the world.

Following the notation of Klaassen (2002), let $r_{k,t+1}^{n}$ be the return on asset class $k\left(k=1,...,K\right)$ between time t and $t+1$ if state $n\left(n=1,...,N\right)$ of the world materialises at time $t+1$. Klaassen (2002) mentions a useful result, that if the set of equations

$$\sum_{n=1}^{N} v_n\left(1+r_{k,t+1}^{n}\right)=1 \text{ for all } k=1,...,K,$$

has a strictly positive solution v_n for all $n\left(n=1,...,N\right)$, then no arbitrage opportunities of the first or second type exist (also see Ingersoll, 1987). Taking $r_{\tau,t+1}^{n}$ to be the return on a zero-coupon bond with maturity $k=\tau$, then

$$1+r_{\tau,t+1}^{n}=\frac{P_{t+1}^{n}\left(\tau-1\right)}{P_t\left(\tau\right)}$$

where

$$P_t\left(\tau\right)=e^{-\tau y_t\left(\tau\right)}$$

is the price at time *t* of a zero-coupon bond with maturity τ. Thus if the set of equations

$$\sum_{n=1}^{N} v_n e^{-(\tau-1)y_{t+1}^{n}(\tau-1)}=e^{-\tau y_t(\tau)} \text{ for all maturities } \tau,$$

has strictly positive solution v_n for all $n\left(n=1,...,N\right)$, then no arbitrage opportunities of the first or second type exist in our yield curve scenarios.

The class of arbitrage-free affine dynamic term structure models that Christensen et al. (2007) and Christensen et al. (2008) derives, for the Nelson-Siegel family of yield curves, differs only in the inclusion of a additional yield-adjustment term which depends only on the maturity of the zero-coupon bond. As this term is dependent on the maturity of the bond, it can be seen as a shift in the slope of the yield curve. Now let

$$v_n=\frac{e^{-y_t(1)}}{N} \text{ for all } n\left(n=1,...,N\right),$$

then, if we can find yield curve shifts $c_{t+1}\left(\tau\right)$ such that

$$\frac{1}{N}\sum_{n=1}^{N} e^{-(\tau-1)\left(y_{t+1}^{n}(\tau-1)+c_{t+1}(\tau)\right)}=\frac{e^{-\tau y_t(\tau)}}{e^{-y_t(1)}} \text{ for all maturities } \tau,$$

no arbitrage opportunities exists in the yield curve scenarios. Thus, if the present value of the expected price of a zero-coupon bond with maturity τ equals the current value of a zero-coupon bond with maturity τ, for all maturities, no arbitrage opportunities exists in the yield curve scenarios (this is consistent with no-arbitrage literature).

Given the small size of branching factors of the scenario trees generated it may not be possible to find realistic solutions to the yield curve shifts $c_{t+1}\left(\tau\right)$. Thus to eliminate most of the arbitrage opportunities in the scenario trees we propose the following algorithm:

Step 1: At the root node create a group of N scenarios. Generate all the scenarios using Monte Carlo simulation and the four-factor yields-macro model (as for the scenario tree). Each scenario is equally likely and consists of T sequential yield curves.

Step 2: At each branching time of the scenario tree calculate the average of the N generated scenarios (at the root node the current yield curve is used).

Step 3: Then for each average yield curves and the corresponding one-period ahead scenarios solve

$$\frac{1}{N}\sum_{n=1}^{N} e^{-(\tau-1)\left(y_{t+1}^{n}(\tau-1)+c_{t+1}(\tau)\right)} = \frac{e^{-\tau y_t(\tau)}}{e^{-y_t(1)}}$$

for all maturities, to obtain the yield curve shifts $c_{t+1}(\tau)$.

Step 4: Add the amount $c_{t+1}(\tau)$ to the original scenario tree yield curves.

The described method removes most of the arbitrage opportunities in the scenario tree with a few opportunities left in sub-trees. For scenario trees with a short horizon all opportunities may be removed. We judge this reduction of arbitrage opportunities as sufficient, since portfolio constraints in the optimisation problem, such as the restriction of short-selling and the inclusion of transaction costs, will eliminate or minimise the effect of the remaining arbitrage opportunities.

4.3 BACK-TESTING

To test our scenario generation methodology we implemented the *minimum guarantee multistage stochastic optimisation problem* described in Dempster et al. (2006). Dempster et al. (2006) proposed an asset and liability management framework and give numerical results for a simple example of a closed-end guaranteed fund where no contributions are allowed after the initial cash outlay. They demonstrate the design of investment products with a guaranteed minimum rate of return focusing on the liability side of the product. We use our scenario generation approach to generate the input scenarios for the optimisation problem. The four-factor yields-macro model is fitted to market data up to a initial decision time t and scenario trees are generated from time t to some chosen horizon $t+T$. The optimal first stage/root node decision are then implemented at time t and we measure the success of the portfolio implementation by its performance with historical data up to time $t+1$. This whole procedure is rolled forward for T trading times. At each decision time t, the parameters of the four-factor yields-macro model are re-estimated using the historical data up to and including time t.

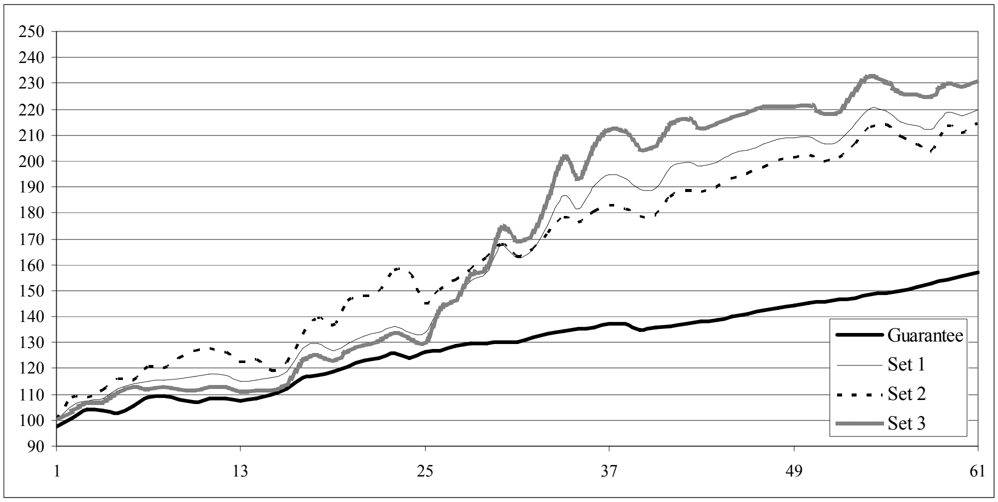

Figure 13. Scenario back-test results.

Table 13. Tree structure for different back-tests

Year	Set 1	Set 2	Set 3
April 03	5.5.5.5.5 = 3125	13.4.4.4.4=3328	200.2.2.2.2 = 3200
April 04	8.8.8.8 = 4096	15.6.6.6 = 3240	400.2.2.2 = 3200
April 05	15.15.15 = 3375	30.10.10 = 3000	400.3.3 = 3600
April 06	56.56 = 3136	160.20 = 3200	800.4 = 3200
April 07	3125	3328	3200

Table 14. Portfolio allocation for different scenario sets

Top40	Y5	Y7	Y10	Y17	Y19
0.1644	0.9104	0	0	0	0
0.1529	0.9230	0	0	0	0
0.2065	0.8639	0	0	0	0
0.2073	0.8631	0	0	0	0
0.0197	1.0699	0	0	0	0
0.2088	0.8615	0	0	0	0

We back-test over a period of five years, from April 2003 through to April 2008, and use different tree structures with approximately the same number of scenarios. The tree structures are described in Table 13. Bonds with 5, 7, 10, 15 and 19 year maturities as well as the FTSE/JSE Top 40 index are included in the portfolio. Scenarios for the Top 40 index are generated along with the yield curve by modelling the index with respect to the three macro-economic variables. We minimise the expected average shortfall for a 9% annual guarantee with transaction costs.

Figure 13 illustrates back-testing results for all three scenario sets. The results are consistent with those in Dempster et al. (2006). The model performs well staying above the guarantee at all times although the system involves the inclusion of truncation cost which puts downward pressure on the portfolio wealth.

In Table 14 we present the first stage optimal portfolio allocations for several different sets of scenario. These sets were all generated with a tree-string of 5.5.5.5.5 from the same data. The back-testing experiments on different scenario sets yield relatively the same first stage portfolio allocations, indicating the stability of the scenario generation.

5. CONCLUSION

This paper considers the estimation and characterisation of the South African term structure with respect to macro-economic variables and its use in scenario generation for fixed income portfolios. We have estimated a yield curve model that incorporates four yield curve factors (level, slope and two curvature factors) and macro-economic variables (real activity, inflation and the stance of monetary policy). The estimated model fits the term structure reasonably well in-sample as shown in the results. The model also performs reasonably well in out-of-sample forecasting. We have shown that better performance can be realised by including the investors expected view on the repo-rate.

We also proposed a parallel simulation approach for yield curve scenario tree generation. The procedure is tested and the performance is measured by out-of-sample back-testing in terms of the value of a fixed income portfolio optimization problem described in the literature. The results demonstrate a reasonably sound way to generate stable yield curve scenario trees. We also discuss the existence of arbitrage in the scenario trees and propose a method to reduce arbitrage opportunities.

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REFERENCES

Ang, A & Piazzesi, M (2003). A no-arbitrage vector autoregression of term structure dynamics with macroeconomic and latent variables. *Journal of Monetary Economics* 50, 745-787

Bank for International Settlements (1999). Zero-coupon yield curves: technical documentation. Bank for International Settlements, Switzerland

Bond exchange of South Africa (2003a). An introduction to the BEASSA zero coupon yield curves. Bond exchange of South Africa, Johannesburg

Bond exchange of South Africa (2003b). The BEASSA zero coupon yield curves: Technical specifications. Bond exchange of South Africa, Johannesburg

Brousseau, V (2002). The functional form of yield curves. Working Paper 148, European Central Bank

Christensen, JH, Diebold, FX & Rudebusch, GD (2007). The affine arbitrage-free class of Nelson-Siegel term structure models. NBER working paper, No. 13611, November

Christensen, JH, Diebold, FX & Rudebusch, GD (2008). An arbitrage-free generalized Nelson-Siegel term structure model. Reserve Bank of San Francisco, Working Paper 08-07

Chueh, YCM (2002). Efficient stochastic modelling for large and consolidated insurance business: Interest rate sampling algorithms. North American Actuarial Journal 6, 88-103

Dempster, MAH, Germano, M, Madova, EA, Rietbergen, MI, Sandrini, F & Scrowston, M (2006). Managing guarantees, *The Journal of Portfolio Management* 32, 51-61

Dewachter, H & Lyrio, M (2002). Macroeconomic factors in the term structure of interest rates. Manuscript, Catholic University Leuven, Erasmus University, Rotterdam

Diebold, FX, Li, C (2006). Forecasting the term structure of government bond yields. *Journal of Econometrics* 130, 337-364

Diebold, FX, Rudebusch, GD & Aruoba, SB (2006). The macroeconomy and the yield curve: a dynamic latent factor approach. *Journal of Econometrics* 131, 309–338

Duffee, GR (2002). Term premia and interest rate forecasts in affine models. *Journal of Finance* 57, 405-443

Dupacova, J, Consigli, G & Wallace, SW (2000). Scenarios for multistage stochastic programs. *Annals of Operations Research* 100, 25-53

Filipović, D (1999). A note on the Nelson-Siegel family. *Mathematical Finance* 9, 349-359

Gülpinar, N, Rustem, B & Settergren, R (2004). Simulation and optimization approaches to scenario tree generation. *Journal of Economic Dynamics & Control* 28, 1291-1315

Hamilton, JD, (1994). Time series analysis. Princeton University Press, Princeton, New Jersey

Harvey, AC (1989). Forecasting, structural time series models and the Kalman filter. Cambridge University Press, Cambridge

Hördahl, P, Tristani, O & Vestin, D (2002). A joint econometric model of macroeconomic and term structure dynamics. Manuscript, European Central Bank

Ingersoll, Jr., JE (1987). Theory of Financial Decision Making. Rowman & Littlefield, Totowa, NJ

Kalman, RE, (1960). A new approach to linear filtering and prediction problems. *Journal of Basic Engineering, Transactions ASME*, Series D 83, 35-45

Klaassen, P (2002). Comment on "Generating scenario trees for multistage decision problems". *Management Science* 48, 1512-1516

Kozicki, S & Tinsley, PA (2001). Shifting endpoints in the term structure of interest rates. *Journal of Monetary Economics* 47, 613-652

Lemke, W (2006). Term structure modelling and estimation in a state space framework. Springer-Verlag, Berlin

Nelson, CR & Siegel, AF (1987). Parsimonious modeling of yield curves. *Journal of Business* 60, 473–489

Rudebusch, GD & Wu, T (2003). A macro-finance model of the term structure, monetary policy, and the economy. Manuscript, Federal Reserve Bank of San Francisco

Svensson, LEO (1994). Estimating and interpreting forward rates: Sweden 1992-4. *CEPR Discussion Paper Series* No. 1051, October

Wu, T (2002). Monetary policy and the slope factors in empirical term structure estimations. Federal Reserve Bank of San Francisco, Working Paper 02-07