MODELLING THE MARKET IN A RISK-AVERSE WORLD: THE CASE OF SOUTH AFRICA

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ABSTRACT
In this paper, descriptive models of real returns on the market portfolio of South Africa are developed and analysed. The ‘market portfolio’ is taken to comprise listed equity and government bonds, aggregated in proportion to their market capitalisation from time to time. The models have the attributes that, conditionally on information at the start of a year:

- the real return on the market portfolio during that year is normally distributed; and
- the market price of risk during that year is reasonably greater than zero.

For the purpose of predictive modelling the best of the models considered was found to be a linear function of the risk-free rate. For this purpose it was decided to use ex-ante estimates of expected returns. This led to bias in the observed mean returns, which negates the rational expectations hypothesis. In the light of the literature on the subject, this is considered acceptable for these purposes.

KEYWORDS
Market portfolio, Risk aversion, South Africa, Bias, Rational expectations

1. INTRODUCTION

In this paper, descriptive models of returns on the market portfolio of South Africa are developed and analysed. For the purposes of the paper, ‘returns’ are defined as real annual forces of return. As in Thomson (unpublished a) real returns are used because, in the final analysis, equilibrium must relate to goods and services, not to currencies. The market portfolio is taken to comprise listed equity and government bonds (both conventional and index-linked), aggregated in proportion to their market capitalisation from time to time. The models have the attributes that, conditionally on information at the start of a year:

- the return on the market portfolio during that year is normally distributed; and
- the market price of risk during that year is reasonably greater than zero.

For the purposes of the latter attribute, the market price of risk is taken to be:

$$R_t = \frac{\mu_{M;t} - \delta_{I;t}}{\sigma_{M;t}} \text{ for } t = 1, \ldots, N;$$

where:

$\mu_{M;t} = E\left\{\delta_{M;t} \mid F_{t-1}\right\}$;

$\sigma^2_{M;t} = E\left\{\left(\delta_{M;t} - \mu_{M;t}\right)^2 \mid F_{t-1}\right\}$;

$\delta_{I;t}$ is the real return on a one-year risk-free zero-coupon bond during the year $\left[t-1, t\right]$;

$\delta_{M;t}$ is the real return on the market portfolio during that year; and

F_t is the information at time t, including $\delta_{I;t}$.

The purpose of the development of the descriptive models is to inform the definition of predictive stochastic models for use with the equilibrium models developed in Thomson & Gott (unpublished a, b). (The distinctions between ‘descriptive’, ‘predictive’ and ‘normative’ models follow Thomson (2006a).) In the development of the descriptive models, it is therefore borne in mind that the purpose of estimation is to derive ex-post estimates of ex-ante parameters. The rational expectations hypothesis is applied so far as it is possible to do so. However, where that hypothesis conflicts with this purpose, constraints on or adjustments to the estimates are accepted.

By the same token, the role of $\delta_{I;t}$ in the models is not primarily to explain the variability of $\delta_{M;t}$; other variables might do so better. It is primarily to satisfy the required attributes. Attention is drawn below to instances in which these issues arise. The predictive model envisaged is not intended to constitute ‘the real-world model’ in any unique or logical positivist sense of that concept. It is merely intended to be a reasonable model for the purposes of ex-ante decision-making. For this reason, as in Thomson & Gott (unpublished b), no hypothesis testing is undertaken, and no out-of-sample tests are made.

The requirement that the conditional distribution of the return on the market portfolio during some future year (conditional, that is, on information at the start of that year) is normal does not mean that the unconditional distribution of that return will be normal. If, for example, the return in some future year is a function of another variable whose value will be known at the start of that year, but which is not normally distributed in terms of prior information, then the unconditional distribution of the return will not generally be normal.

Indeed, it is largely the purpose of this paper to explore the use of models in which the unconditional distribution of the return on the market portfolio is not necessarily normal.

It also follows that the market price of risk should not only be reasonably greater than zero in the descriptive model, but should also be so for any reasonable value of $\delta_{I;t}$ that may occur in a predictive model. The interpretation of 'reasonably greater than zero' is amplified in sections 3 and 4 below.

In section 2, relevant literature is reviewed. In section 3 the models are described. In section 4 the parameterisation of the models is presented and the results are discussed. In section 5 the use of the models for predictive purposes is discussed.

A similar exercise has been undertaken by the author using United Kingdom data. The results are reported in Thomson (unpublished).

2. LITERATURE REVIEW

2.1 RISK AVERSION

As pointed out by Merton (1980: 327):

> "… a necessary condition for equilibrium is that the expected return on the market must be greater than the riskless rate.… A sufficient condition for this proposition to obtain is that all investors are strictly risk-averse expected utility maximizers."

For this proposition to obtain, the market price of risk must be positive. While some models of market equilibrium do not rule out a negative market price of risk (e.g. Conrad & Kaul, 1988: 410; Derrig & Orr, 2004: 46), it must be accepted that the long-term financial institutions advised by actuaries (principally life offices and pension funds), effectively being custodians of trust moneys, are risk-averse. These clients are participants in the process of equilibrium formation in the capital market. For actuarial purposes, therefore, the models used by actuaries for advising such clients must assume risk aversion.

Since the publication of the Wilkie (1986) model, numerous stochastic models of returns on assets have been published. Most of these suffer from the drawback that, conditionally on information at the start of a period, they may produce negative market prices of risk during that period. Among the models that may exhibit this phenomenon (not all of which are published in detail), particularly in the case of equities, are:

- the Wilkie (1986, 1995b) model for the United Kingdom (also calibrated for other countries);
- the Carter (1991) model for Australia;
- the Dyson & Exley (1995) model for the U.K.;
- the Thomson (1996) model for South Africa;
- the Harris (1997) model for Australia;
- the CAP:Link scenario generation system (Mulvey & Thorlacius, 1998);
- the Boender, Van Aalst & Heemskerk (1998) model for the Netherlands;
- the Whitten & Thomas (1999) model for the U.K.;
- the TY model for the U.K. (Yakoubov, Teeger & Duval, 1999); and
- the Hilli et al (2007) model for Finland.

The Hibbert, Mowbray & Turnbull (unpublished) model for the U.K. avoids this problem.

2.2 THE MARKET PORTFOLIO

None of the models listed in the preceding section includes a model of the return on the market portfolio. While they do produce models of major constituents of the market portfolio, their aggregation into a model of the market portfolio would require a model of the composition of that portfolio. The advantage in the explicit modelling of a market-portfolio proxy is that it permits the equilibrium modelling of the various asset categories.

As stated above, in this paper the market portfolio is taken to comprise listed equity and government bonds, aggregated in proportion to their market capitalisation from time to time. As Roll (1977) points out, a model of the market portfolio should include not only equities and bonds, but also all other capital assets, including non-traded assets such as human capital. While this would indeed be required for a true descriptive model, the requirements of a normative model for decision-making purposes are less exacting. Again it must be appreciated that the institutional clients of actuaries invest in a market of traded assets and participate in the process of equilibrium formation within that market. Eun (1994) analyses the capital-asset pricing model (CAPM) into the observable and latent portfolios comprising the market portfolio. He finds that, if the correlation between the two is positive, then, for the observable constituent of the market portfolio, the securities market line has a higher intercept than the risk-free rate. The excess is proportional to the risk premium on the latent constituent. If, however, equilibrium occurs between participants excluded from the latent portfolio, then, for those participants, the intercept must revert to the risk-free rate. This can be accounted for only if the homogeneity of expectations differs as between those participants and others.

The Thomson & Gott (unpublished a) model for South Africa avoids the problem of negative market prices of risk and it includes a model of the market portfolio. However, the specification of the model of the market portfolio in that article was tentative. The exploration of alternative market models was left to further research, which is the subject of this paper.

2.3 BIAS AND RATIONAL EXPECTATIONS

The approach adopted in this paper admits the possibility of bias in conditional expected returns on the market portfolio.

Merton (1980: 125–6) observed that, while substantial effort had been expended on the estimation of the volatilities of returns, little work had been done on expected returns. He suggested that this was due to the relative difficulty of estimating expected returns. However, as Derrig & Orr (*op. cit.*: 46), Campbell (2000: 1522) and Grant & Quiggin (2006) point out, since Mehra & Prescott's (1985) exploration of the 'equity risk-premium puzzle', there have been numerous articles reviewing the expected returns on equity. Conrad & Kaul (*op. cit.*) postulate an autoregressive process for conditional expected return, but their model does not exclude negative market prices of risk. Fama & French (1989) find that expected returns follow a business-cycle pattern and contain a risk premium that is related to longer-term aspects of business conditions. Derrig & Orr (*op. cit.*) document numerous different approaches to the estimation of the equity risk premium, with widely differing results. Wilkie (1995a) contributes yet another. Thomson (2006b) suggests that reference to the equity risk-premium 'puzzle' suggests a paradigmatic metanarrative that needs to be deconstructed.

An often unstated assumption underlying the calibration of stochastic models of returns on assets is that the rational expectations hypothesis (REH) (Muth, 1960) holds. While some authors (e.g. Thomson, 1996: 798–9) have cautioned prospective users that their descriptive models may not be appropriate for predictive purposes, the calibration of those models to ex-post observations suggests that, in the absence of information to the contrary, those observations are unbiased estimates of the corresponding ex-ante values.

Numerous studies (Cuthbertson, 1996: 116–201) show that, on certain assumptions, for certain markets at certain times, the REH may be rejected. While many of these relate to short-term effects or to individual shares relative to the market, some of them (e.g. Shiller, 1981; LeRoy & Porter, 1981) are of importance in the long-term modelling of the market. Even in those cases, it has been shown (e.g. Marsh & Merton, 1986) that, with different assumptions, different conclusions may be drawn and Fama (1991: 1586) argues that they do not necessarily reject the REH. Nevertheless, the REH remains questionable. As Cuthbertson (*ibid.*: 97) points out, tests of the REH that rely on an assumed model such as the capital-asset pricing model (CAPM) involve joint assumptions; rejection does not necessarily imply rejection of the REH. Conversely, however, they would not necessarily imply rejection of the CAPM.

As Roll & Ross (1994) pointed out:

> "… a decade of empirical studies [had] reported little evidence of a significant cross-sectional relation between average returns and betas."

A possible explanation, they suggested, is that market-portfolio proxies are mean–variance inefficient. Another possible explanation is that the REH does not apply. They refer to the phenomenon as a 'puzzle'; like the equity risk-premium puzzle, this begs the question whether the paradigm presupposed by the REH is true.

3. MODELS OF THE MARKET PORTFOLIO

In Thomson & Gott (unpublished a) a simple model of the return on the market portfolio was adopted, without consideration of more complex but possibly better models. In this section four models are considered: the basic model used in that article (a linear function of the risk-free rate), a Markov regime-switching model, an exponential autoregressive (AR) model and an autoregressive conditional heteroskedasticity (ARCH) model. These models are defined below.

The data used are the returns on the South African market-portfolio proxy and on the one-year risk-free zero-coupon bond for the period from 1987 to 2007. As noted above the market-portfolio proxy comprises South African listed equity and government bonds (both conventional and index-linked). These were the same data as used in Thomson & Gott (unpublished a), except that data for the two years 1987 and 1988 were not used in that paper, nor were those for 2006 and 2007. The reason for the exclusion of 1987 and 1988 from Thomson & Gott (*ibid.*) was that, in those years, the formula for the risk-free rate (viz. the one-year conventional bond yield less the inflation rate) gave negative values, which conflicted with the requirement of arbitrage-freedom. In this paper the formula has been subjected to a minimum of zero. While this may introduce some bias, that bias is considered acceptable in the light of the discussion in section 2.3. The data for 2006 and 2007 have become available since the date of that paper.

As mentioned in section 2, the market portfolio should include all assets in which the actuary's client may invest. A notable absence is fixed property. Since many fixed properties are owned by listed companies, it would be necessary to avoid the double-counting involved. Corporate debt should also be included. Until such time as the necessary data are available, the portfolio used in this paper is an approximation to the best proxy available.

The data set is small, comprising only 21 values of each of the variables. It would have been possible to use quarterly data, but for the purpose of annual decision-making that would be of questionable value (Thomson, 1996). As Merton (*op. cit.*) points out, the precision of the estimate of expected returns depends on the total length of calendar time, rather than on the number of observations per se. On the other hand, in a rapidly changing world, it is questionable whether long data sets are relevant to the future.

In view of the small data set, particular care needs to be taken to avoid the treatment of spurious or fortuitous relationships as important. Also, in specifying models involving autoregressive effects, long lags should not be considered. If such a lag is of greater significance than a shorter lag, the effect would have to be regarded as fortuitous. In this paper only one-year lags are considered.

Since, as explained above, it is not intended that any predictive model based on the descriptive models developed in this paper is uniquely valid, it is considered better to retain reasonable uncertainty in the model than to achieve high levels of likelihood based on fortuitous relationships.

3.1 THE BASIC MODEL

As explained in Thomson & Gott (unpublished a), $\mu_{M;t}$ cannot be modelled as a constant because this would result in negative risk premiums from time to time. Instead, as in that paper, we may model $\delta_{M;t}$ as:

$$\delta_{M;t} = g\delta_{I;t} + h + \sigma_M \varepsilon_t ;$$

where:

$\varepsilon_t \sim N(0,1)$;

$\mathrm{cov}\{\varepsilon_t, \varepsilon_s\} = 0$ for $s \neq t$;

$g \geq g* \geq 1$; and

$h \geq h* \geq 0$;

so that:

$$\mu_{M;t} = g\delta_{I;t} + h .$$

The purpose of introducing the constants g* and h* is to establish lower bounds for g and h respectively. In order to avoid negative market prices of risk, we may take g* = 1 and h* = 0. These are referred to below as the 'basic constraints'. However, the purpose of setting g* and h* greater than or equal to 0 is to ensure not merely that the market price of risk is non-negative, but also, as explained in section 1, that it is reasonably greater than 0. For this purpose it is required either that $g^* = 1,2$ and $h^* = 0$ or that $g^* = 1$ and $h^* = 0,01$. (The concept 'reasonably greater' is necessarily arbitrary.) These are referred to below as the 'required constraints'.

In Thomson & Gott (unpublished a) it was found that h was not significant at the 95% level. With h = 0 an estimate of g = 1,7 was obtained. The 95% confidence limits of g were 1 and 3,1, so that this estimate was not reliable. It was nevertheless used in that paper for the purposes of illustration.

3.2 THE REGIME-SWITCHING MODEL

Another possible approach would be to use a Markov regime-switching model (Hamilton, 1989), with a similar structure in each regime, i.e.:

$$\delta_{M;t} = g_{S_t}\delta_{I;t} + h_{S_t} + \sigma_{S_t}\varepsilon_t ;$$

where:

$S_t \in \{0,1\}$;

$\Pr\{S_t = 0 \mid S_{t-1} = 0\} = p_{00}$;

$\Pr\left\{S_t = 1 \mid S_{t-1} = 0\right\} = p_{01} = 1 - p_{00}$;
$\Pr\left\{S_t = 0 \mid S_{t-1} = 1\right\} = p_{10}$;
$\Pr\left\{S_t = 1 \mid S_{t-1} = 1\right\} = p_{11} = 1 - p_{10}$;

and $\varepsilon_t \sim N(0,1)$ is serially independent, so that, conditionally on information at time $t-1$:

$$\delta_{M;t} \sim N\left(\mu_{M;t}, \sigma^2_{M;t}\right);$$

where:

$\mu_{M;t} = g_{S_t}\delta_{I;t} + h_{S_t}$;
$\sigma_{M;t} = \sigma_{S_t}$;
$g_s \geq g_s^* \geq 0$; and
$h_s \geq h_s^* \geq 1$.

As for the basic model, it is required, for each *s*, either that $g_s^* = 1{,}2$ and $h_s^* = 0$ or that $g_s^* = 1$ and $h_s^* = 0{,}01$.

As mentioned in section 1 above, it is required that, conditionally on information at the start of a year, the return on the market portfolio during that year be normally distributed. In order to accommodate this requirement it is assumed that F_{t-1} includes S_t; i.e. that the regime is known at the start of the year. It is for this reason that the parameters must satisfy the required constraints in each regime; otherwise the distribution of the return would have a mixture density.

3.3 THE EXPONENTIAL AR MODEL

$\mu_{M;t}$ can also not be modelled as a linear autoregressive moving-average (ARMA) time series because this would also result in negative risk premiums from time to time. However, $\delta_{M;t}$ may for example be modelled as:

$$\delta_{M;t} = g\delta_{I;t} + h\exp\left\{\alpha\left(\delta_{M;t-1} - g\delta_{I;t-1}\right)\right\} + \sigma_M\varepsilon_t;$$

where:

$g \geq g* \geq 1$;
$h \geq h* \geq 0$; and
$\varepsilon_t \sim N(0,1)$ is serially independent;

so that, conditionally on information at time $t - 1$:

$$\delta_{M;t} \sim N\left(\mu_{M;t}, \sigma^2_M\right);$$

where:

$\mu_{M;t} = g\delta_{I;t} + h\exp\left\{\alpha\left(\delta_{M;t-1} - g\delta_{I;t-1}\right)\right\}$.

Again it is required either that $g^* = 1{,}2$ and $h^* = 0$ or that $g^* = 1$ and $h^* = 0{,}01$. However, if $h = 0$, the model reduces to the basic model, so the first constraint must be that $g^* = 1{,}2$ and $h^* > 0$.

This model is referred to in this paper as the 'exponential AR model'.

3.4 THE ARCH MODEL

A fourth possibility is to include ARCH effects (Engle, 1982). $\delta_{M;t}$ may, for example, be modelled as:

$$\delta_{M;t} = g\delta_{I;t} + h + z_t ;$$

where:

$z_t = \sigma_t \varepsilon_t$;

$\sigma_t^2 = a + bz_{t-1}^2$; and

$\varepsilon_t \sim N(0,1)$ is serially independent.

Here it is required that $a > 0$ and $b > 0$; otherwise σ_t^2 may be negative.

3.5 GENERAL REMARKS

It may be noted that the basic model is a particular case of each of the other models. The use of the latter models must therefore be justified in terms of their additional descriptive value.

For each model, the likelihood function of the model and the maximum-likelihood estimates of the parameters were determined, as well as the confidence limits of those estimates.

For the purposes of comparison of the descriptive value of the respective models, but subject to the required attributes, the Akaike information criterion (AIC) was used, viz.:

$$A = 2k - 2l ;$$

where:

k is the number of parameters;

$l = \ln(L)$; and

L is the likelihood of the observed values. (Akaike, 1974).

For each model considered (or, in the case of the regime-switching model, for each regime), the mean market price of risk was calculated, viz.:

$$R = \frac{\hat{\mu}_M - \bar{\delta}_I}{\hat{\sigma}_M} ;$$

where:

$\hat{\mu}_M = g\bar{\delta}_I + h$; and

$\bar{\delta}_I = \frac{1}{N}\sum_{t=1}^{N} \delta_{I;t}$.

Also, the implied bias in the mean was calculated, viz.:

$$B = \bar{\delta}_M - \hat{\mu}_M ;$$

where:

$\bar{\delta}_M = \frac{1}{N}\sum_{t=1}^{N} \delta_{M;t}$.

Finally, a Q–Q plot (Wilk & Gnanadesikan, 1968) was produced.

4. THE PARAMATERISATION OF THE MODELS

In this section the parameterisation of each of the models is presented in turn, and the results are discussed. The results are then compared between the models considered.

4.1 THE BASIC MODEL

For the basic model the likelihood function and the estimates and confidence limits of the parameters were obtained in closed form following (e.g.) Hocking (1996: 136–45) (see section A.1 in the appendix). Table 1 shows the results of the parameterisation. As shown in that table, all three cases satisfy the required constraints.

It may be noted that, in relation to their respective values, the confidence intervals of the parameters are wide, particularly in the cases of *g* and *h*, though they have narrowed slightly in comparison with Thomson (unpublished a) because of the extra data. This leaves considerable scope for discretion in the determination of parameters for the predictive use of the model. The AIC is lowest for $h = 0$ and, for the purposes of this paper, that case is adopted.

Table 1. Parameterisation of the basic model

Parameter	Details	constraints		
		basic	$g = 1$	$h = 0$
g	estimate	1,59		1,76
	confidence limits	1; 3,8		1; 2,9
h	estimate	0,012	0,039	
	confidence limits	0; 0,14	0; 0,11	
σ_M	estimate	0,163	0,160	0,159
	confidence limits	0,11; 0,21	0,11; 0,21	0,11; 0,20
k		3	2	2
l		9,28	9,15	9,30
A		–12,56	–14,30	–14,60
R		0,24	0,24	0,22
B		0	0	0,004

Figure 1 plots the observed values, estimates and confidence limits of $\delta_{M;t}$ against $\delta_{I;t}$ for the basic model. Figure 2 shows the same information in time-series form.

It is clear that $\delta_{I;t}$ explains very little of the variability in $\delta_{M;t}$. However, as explained above, this is not the point; the purpose of including $\delta_{I;t}$ is not to explain the variability in $\delta_{M;t}$, but to ensure that the market price of risk is positive.

For the purposes of the Q–Q plot, the set $\left\{ z_t = \delta_{M;t} - g\delta_{I;t} + h \mid t = 1,\ldots,N \right\}$ was ordered to give $\left\{ z^{(r)} \mid r = 1,\ldots,N \right\}$ such that:

$$z^{(r)} \geq z^{(r-1)} \text{ for } r = 2,\ldots,N .$$

The Q–Q plot was then defined by the points:

$$\left(\Phi^{-1}\left(y^{(r)} \right), z^{(r)} \right);$$

where:

$$y^{(r)} = \frac{r - \frac{1}{2}}{N}; \text{ and} \qquad (1)$$

$\Phi(\bullet)$ is the distribution function of the normal variable with mean 0 and standard deviation σ_M.

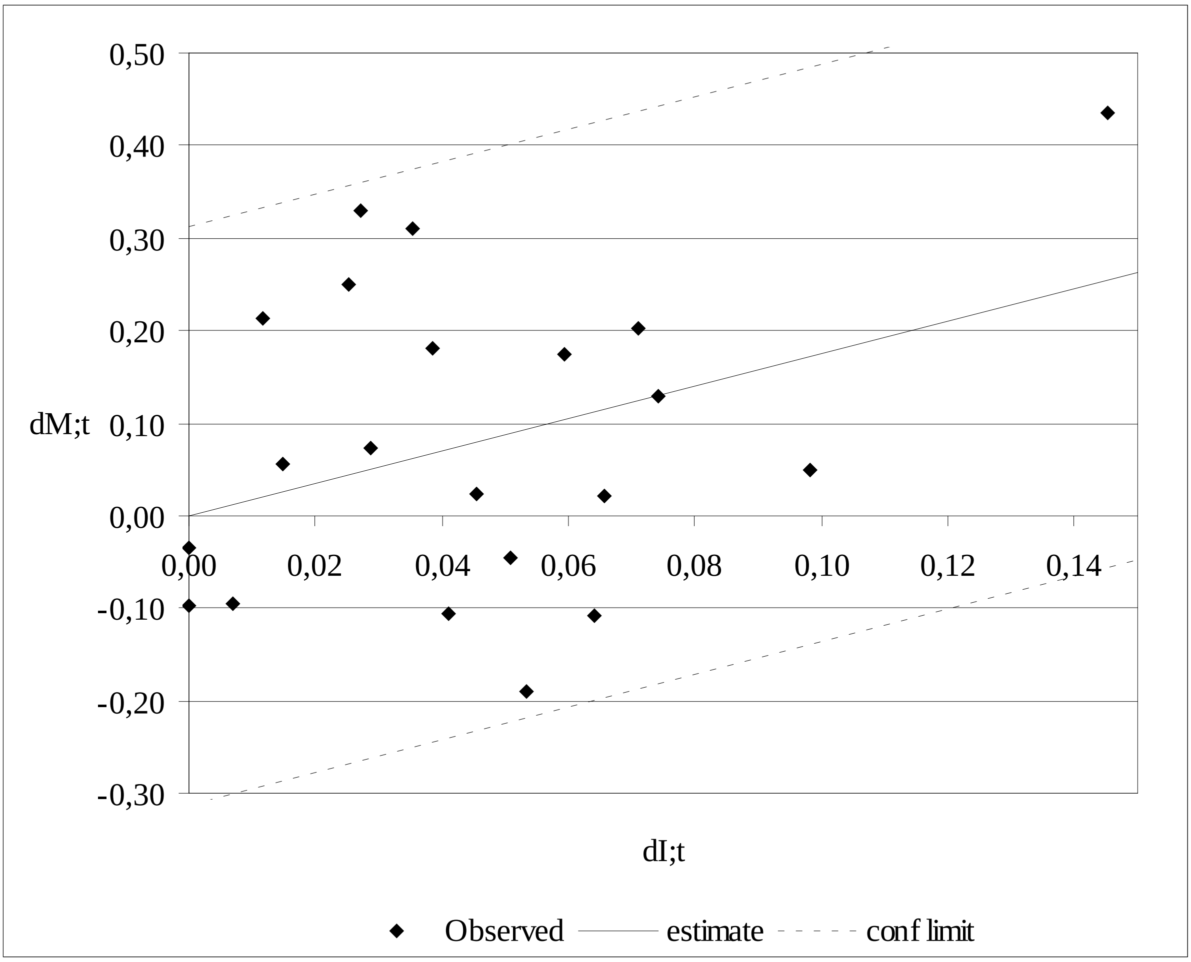

Figure 1. Basic model: $\delta_{M;t}$ against $\delta_{I;t}$

In equation (1) the numerator and denominator represent the number of observations less than $z^{(r)}$ and $z^{(N)}$ respectively, the second term in the numerator being an adjustment for symmetry.

The Q–Q plot of the basic model is shown in Figure 3. In view of the small data set, this plot gives credible results.

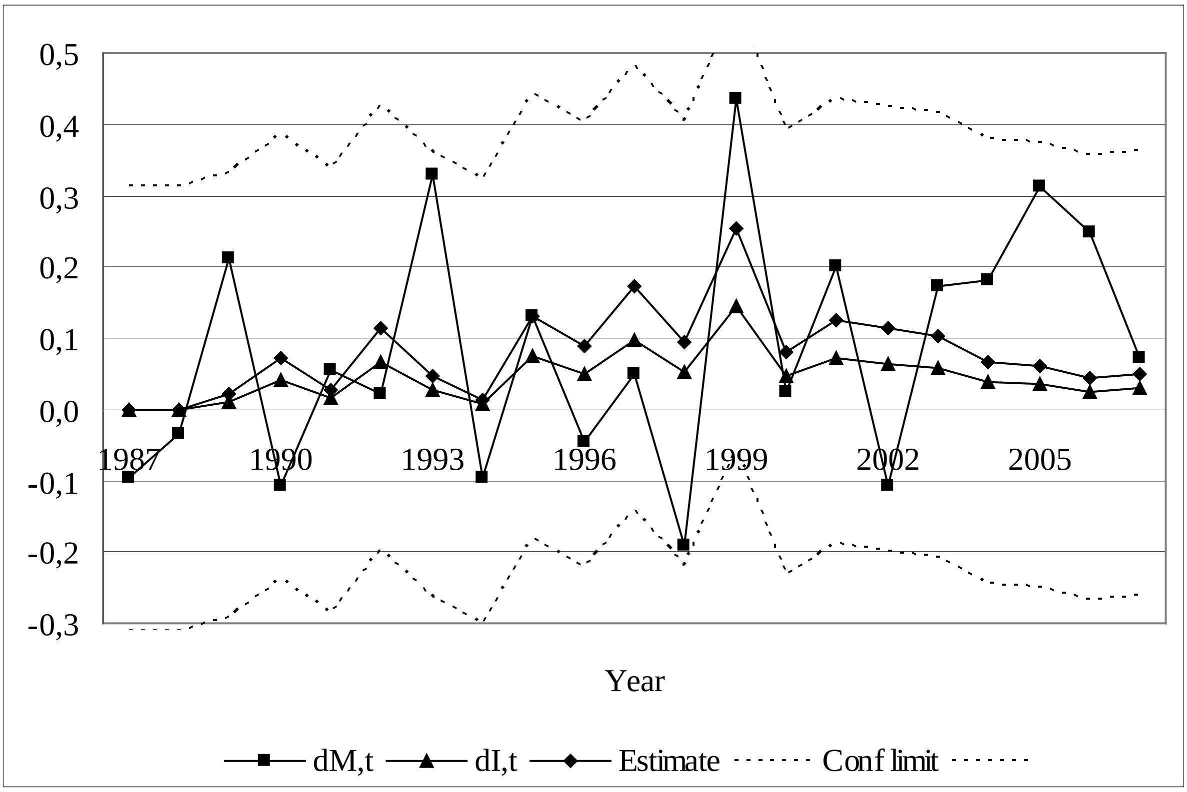

Figure 2. Basic model: time series

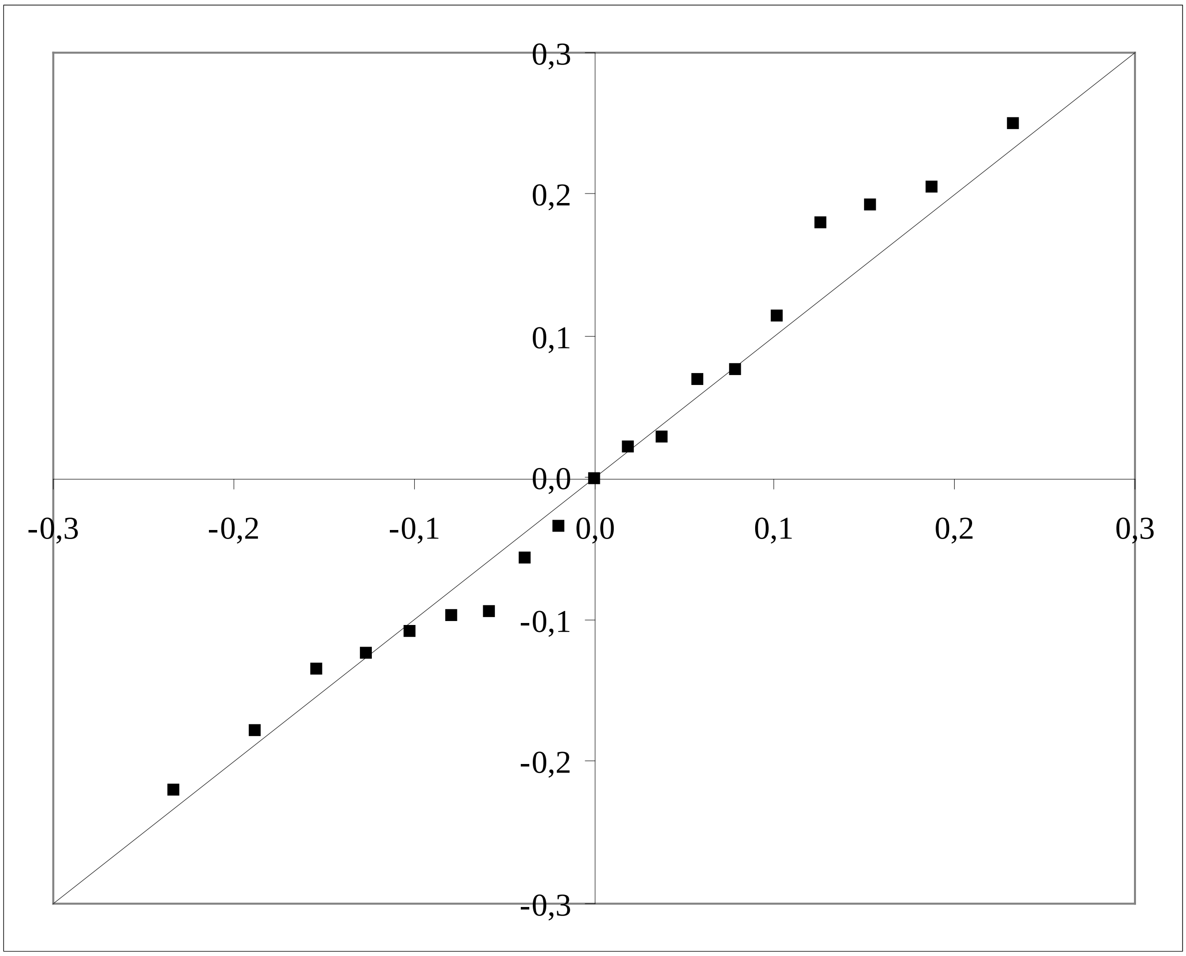

Figure 3. Basic model: Q-Q plot

4.2 THE REGIME-SWITCHING MODEL

For the regime-switching model the likelihood function was found following Hamilton (1989) (see section A.2 in the appendix.) The maximum likelihood was found by means of the Nelder–Mead algorithm (Nelder & Mead, 1965). The results are shown in Table 2. Confidence limits of the parameters were not calculated.

It may be noted from Table 2 that σ_0 is extremely low. This is reflected in the narrow confidence limits of regime 0 in Figure 4, which shows the observed values, estimates and confidence limits of $\delta_{M;t}$ against $\delta_{I;t}$ for the regime-switching model.

Table 2. Parameterisation of the regime-switching model

Parameter	
p_{00}	0
p_{10}	0,24
g_0	2,96
h_0	0
σ_0	0,009
g_1	1,2
h_1	0,007
σ_1	0,162
l	13,25
k	7
A	–12,50
R	0,56
B	–0,051

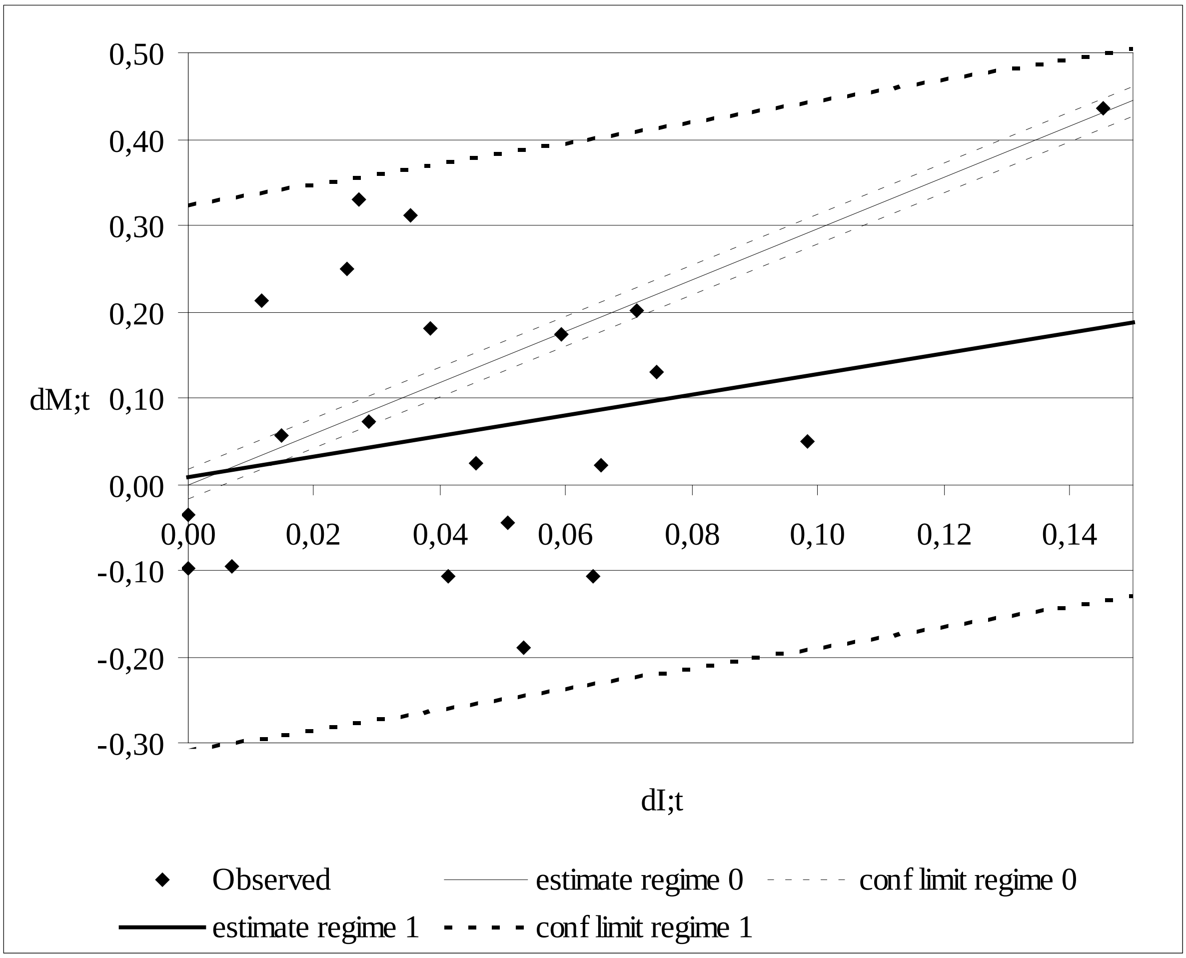

Figure 4. Regime-switching model

It may be noted from Figure 4 that the parameterisation effectively identifies a fortuitous alignment of five points. These are associated with regime 0 with high probabilities, while all other points are associated with regime 1 with even higher probabilities. As mentioned in section 3, in view of the small data set, particular care needs to be taken to avoid the treatment of spurious or fortuitous relationships as important. For this reason, the regime-switching model should not be adopted without further analysis.

4.3 THE EXPONENTIAL AR MODEL

For the exponential AR model the likelihood function was obtained in closed form as shown in section A.3 in the appendix, but it was not possible to obtain the estimates of the parameters in closed form. As for the regime-switching model, the estimates were obtained by means of the Nelder–Mead algorithm.

Table 2 shows the results of the parameterisation. Because the parameterisation is based on 20 observations instead of 21, the value of the log-likelihood has been multiplied by 21/20. This adjustment is not necessarily accurate, as the contribution of year 1 does not necessarily correspond to the average log-likelihood. As for the basic model, in relation to their respective values, the confidence intervals of the parameters other than σ_M are quite wide.

Figure 5 plots the observed values, estimates and confidence limits of $\delta_{M;t}$ in time-series form.

Table 3. Parameterisation of the exponential AR model

Parameter	Details	
α	estimate	–13,75
	confidence limits	–16,9; 3,0
g	estimate	1
	confidence limits	1; 2,1
h	estimate	0,01
	confidence limits	0,01; 0,06
σ_M	estimate	0,155
	confidence limits	0,11; 0,22
k		2
l		10,34
A		–16,68
R		0,23
B		0,003

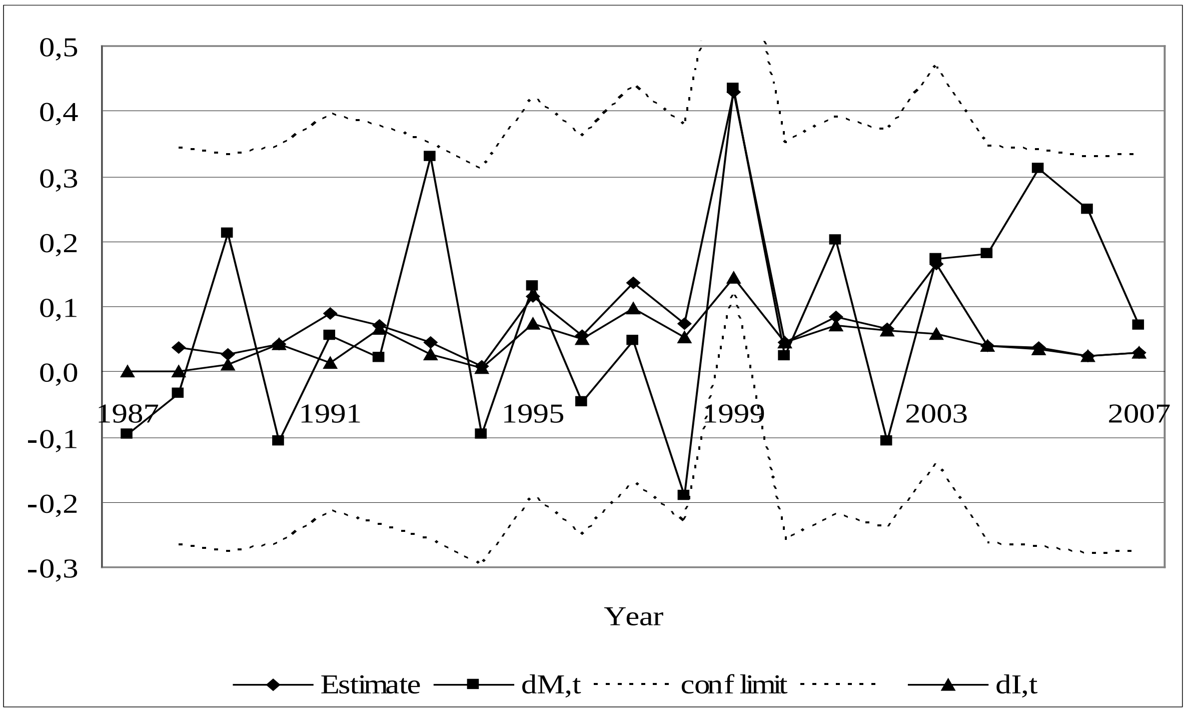

Figure 5. Exponential AR model time series

A Q–Q plot is shown in Figure 6. It is not as good as that of the basic model.

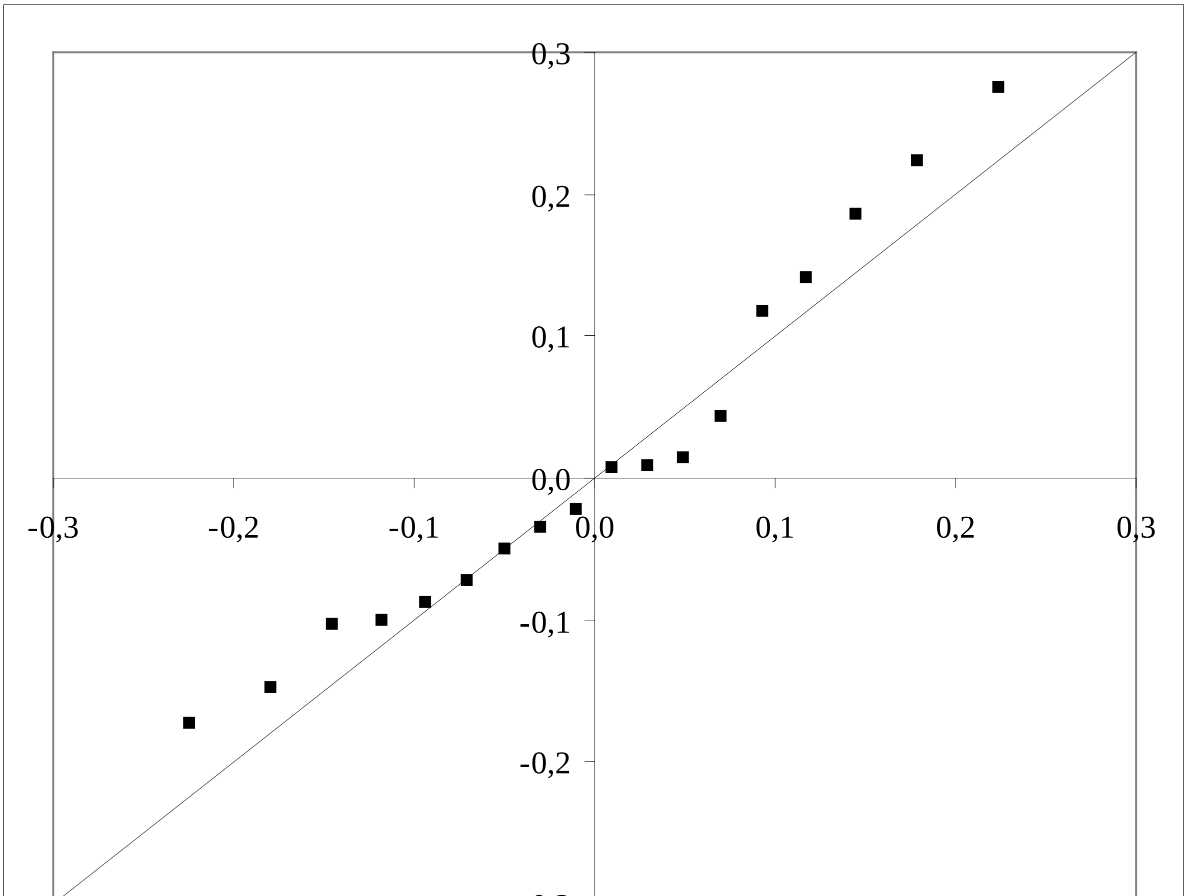

Figure 6. Exponential AR model Q-Q plot

4.4 THE ARCH MODEL

For the ARCH model the maximum-likelihood estimates of g and h are the same as those for the basic model. The likelihood function was obtained in closed form as shown in section A.4 of the appendix and the Nelder–Mead algorithm was used to determine the maximum-likelihood estimates of the parameters. The 95% confidence limits of the estimates were again determined by simulation.

Table 4 shows the results of the parameterisation of the ARCH model. From that table it may be noted that the estimate of b is zero, the minimum constraint. This means that the model reduces to the basic model, with:

$$\sigma_M^2 = (0{,}159)^2 = 0{,}025 = a\ .$$

The ARCH model is therefore not further considered.

Table 4. Parameterisation of the ARCH model

Parameter	Details	
g	estimate	1,76
	confidence limits	1; 2,9
h		0
a	estimate	0,025
	confidence limits	0,021; 0,029
b	estimate	0
	confidence limits	0; 0,04
k		2
l		9,30
A		–14,60
R		0,22
B		0,004

4.5 SUMMARY

Table 5 summarises the selection criteria of the descriptive models. According to the AIC, the exponential AR model is considerably superior to the other models.

Table 5. Summary of selection criteria

	Model		
Criterion	basic	regime-switching	exponential AR
A	–14,60	–12,50	–16,68
R	0,22	0,56	0,23
B	0,004	–0,051	0,003

On the basis of Table 5, it would appear that the exponential AR model should be selected. However, as shown in Figures 2 and 5, the Q–Q plot of the basic and exponential AR models suggest that the residuals of the former give a more credible fit to a normal distribution, though the difference is not substantial. In view of the small sample size, care must be taken to avoid spurious effects. Further analysis of the likelihoods of the basic and exponential models is therefore necessary.

In the first instance, it may be noted that the formulation of the exponential AR model relaxes the requirement that the excess return on the market must exceed 0,01. This is particularly problematic in the case in which (as in this parameterisation) $g = 1$.

Another effect increasing the likelihood of the exponential AR model is the good fit of the latter in 1999. Indeed, it appears that the optimisation of α is largely to achieve this effect.

Thirdly, as pointed out in section 4.3, the multiplication of the likelihood of the exponential AR model for the purposes of comparison is not accurate. In fact, one of the larger positive errors in the basic model is in year 1—the year that is omitted in the exponential AR model.

Furthermore, it may be noted from Table 3 that the 95% confidence limits of α range from –16,9 to 3,0. If $\alpha = 0$ then the model would reduce to the basic model (with $g = 1$).

While the regime-switching model may be rejected on the grounds of Table 5, the choice as between the basic and exponential AR models is not conclusive. Furthermore, as shown in the next section, it is affected by considerations relating to the use of the model for predictive purposes. The choice is therefore further considered in the next section.

5. THE USE OF THE MODEL FOR PREDICTIVE PURPOSES

As explained in section 1, the purpose of the development of the descriptive models in this paper is to derive ex-post estimates of ex-ante parameters. The reason for this is that, for predictive purposes, ex-ante parameters are required.

The question that arises in using the descriptive model to inform the development of a predictive model, is whether and to what extent the biases of the past will persist in the future. A different way of posing the question is: While the market did not follow the ex-ante estimates of the model in the past, is it reasonable to assume that it will do so in the future? The first presupposes that the ex-ante estimates were wrong, while the second presupposes that the sample observed was fortuitously different from the estimates. The question is particularly pertinent in relation to the regime-switching model, which implied a large bias.

On the basis of the parameterisation of the exponential AR model, the ex-post estimate of μ_M is 0,081.

For predictive purposes an unbiased ex-ante estimate of μ_M is required. This is not necessarily best estimated by ex-post maximum likelihood. As mentioned in section 2, the problem of unbiased estimation of ex-ante expected returns has been addressed by numerous authors. Amongst these are the following estimates of the excess return on equities over the risk-free rate (all expressed as annual rates):

- Wilkie (1995a): 3% long-term median on U.K. equities;
- Derrig & Orr (2004): 4% to 5% on U.S. equities;
- Campbell (unpublished): 3,8% on world equities.

If we take the estimate for equity at 3,8%, this converts to an annual force of 3,7%. Risk premiums on long-term bond returns are generally lower. From the CAPM:

$$\frac{\breve{\mu}_E}{\breve{\mu}_C(20)} = \frac{\hat{\sigma}_{EM}}{\hat{\sigma}_{CM}(20)};$$

where:

$\breve{\mu}_E = 0,037$ is the excess return on equities over the risk-free rate;

$\breve{\mu}_C(20)$ is the excess return on conventional bonds over the risk-free rate;

$\hat{\sigma}_{EM}$ is the covariance between the return on equities and the return on the market; and

$\hat{\sigma}_{CM}(20)$ is the covariance between the return on a 20-year conventional bond and the return on the market.

The 20-year conventional bond is used for simplicity. Index-linked bonds are ignored because of their relatively small market capitalisation. From the data described in Thomson (unpublished a), extended to 2007, and using the definitions in that paper, we obtain:

$\hat{\sigma}_{EM} = 0,032$; and

$\hat{\sigma}_{CM}(20) = 0,005$.

This gives:

$$\breve{\mu}_C(20) = \frac{\breve{\mu}_E \hat{\sigma}_{CM}(20)}{\hat{\sigma}_{EM}} = 0,006.$$

With a weighting of bonds to equities of 0,12:0,88, the excess annual return on the market portfolio may be taken at:

$$0{,}12\breve{\mu}_C(20)+0{,}88\breve{\mu}_E=0{,}033.$$

At the time of writing (September 2008) the yield on long-term South African index-linked gilts is 3,0% a year, convertible half-yearly. If this is taken as indicative of currently expected short-term real interest rates then we can take the ex-ante mean risk-free rate as:

$$\breve{\mu}_I=2\ln\left(1+\frac{0{,}03}{2}\right)=0{,}030;$$

so that the ex-ante expected return on the market portfolio is:

$$\begin{aligned}\breve{\mu}_M&=0{,}030+0{,}033\\&=0{,}063.\end{aligned}$$

This means that the expected value of $\delta_{M;t}$ must be adjusted. As parameterised, the formula of the exponential AR model is:

$$\delta_{M;t}=\delta_{I;t}+0{,}01\exp\left\{-13{,}75\left(\delta_{M;t-1}-g\delta_{I;t-1}\right)\right\}+\sigma_M\varepsilon_t.$$

If either of the factors parameterised as 0,01 and –13,75 in the second term on the right-hand side is reduced, the problem of the relaxation of the constraints discussed in section 4.5 is exacerbated. If the first term on the right-hand side is multiplied by a constant less than unity, or if a constant is deducted from that side, the basic constraints are violated, and there is a non-zero probability that, conditionally on information at the start of year *t*, the expected excess return on the market during that year will be negative.

In the light of the problems discussed in section 4.5 and the further difficulties arising here, it must be accepted that the exponential AR model cannot generally be used for predictive purposes. The basic model must therefore be used. For that purpose the value of *g* must be adjusted.

On the basis of the parameterisation of the basic model, the ex-post estimate of μ_M is 0,080. In order to reduce μ_M to 0,063, we reduce *g* to:

$$1{,}76\,\frac{0{,}063}{0{,}080}=1{,}39\ .$$

Since this satisfies the required constraints and falls within the confidence limits of the original estimate, the latter value is adopted with no further reconsideration of the model.

6. SUMMARY AND CONCLUSION

In this paper, the development of descriptive models of the South African market portfolio is described. The models have the attributes specified in section 1. For the purpose of predictive modelling the best of those models was found to be the basic model. For this purpose it was decided to use ex-ante estimates of expected returns. This implied bias in the observed mean returns, which negates the REH. In the light of the literature on the subject, this is considered acceptable for these purposes.

The model is defined as:

$$\delta_{M;t}=g\delta_{I;t}+\sigma_M\varepsilon_t;$$

where:

$g=1{,}39$;

$\sigma_M=0{,}159$;

and $\varepsilon_t \sim N(0,1)$ is serially independent

The model is suitable for use with the equilibrium model developed in Thomson & Gott (unpublished a) and in the pricing of the liabilities in an incomplete market as proposed by Thomson (2005).

As time goes by it will be necessary to revisit the parameterisation of the model.

ACKNOWLEDGEMENTS

The assistance of Mr D.V. Gott—particularly with regard to the data used—and his peer-review of this paper are acknowledged. The financial assistance of the Actuarial Society of South Africa is also acknowledged. Opinions expressed and conclusions drawn are those of the author and are not to be attributed to the Society.

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APPENDIX
LIKELIHOOD FUNCTIONS AND MAXIMUM-LIKELIHOOD ESTIMATES OF THE MARKET-PORTFOLIO MODELS

A.1 The basic model

The log-likelihood function of the basic model follows that of linear regression (e.g. Hocking, 1996: 136–45), viz.:

$$l(\theta) = -\frac{N}{2}\ln\left(2\pi\sigma_M^2\right) - \frac{1}{2\sigma_M^2}\sum_{t=1}^{N} z_t^2 \ ;$$

where:

$$\theta = \begin{pmatrix} g \\ h \end{pmatrix};$$

$$z_t = \delta_{M;t} - g\delta_{I;t} - h \, .$$

The maximum-likelihood estimates may be derived from the above equations to give:

$$\hat{g} = \frac{S_{IM}}{S_{II}} \text{; and}$$

$$\hat{h} = \overline{\delta}_M - \hat{g}\overline{\delta}_I \ ;$$

where:

$$S_{IM} = \sum_{t=1}^{N}\left(\delta_{I;t} - \overline{\delta}_I\right)\left(\delta_{M;t} - \overline{\delta}_M\right);$$

$$S_{II} = \sum_{t=1}^{N}\left(\delta_{I;t} - \overline{\delta}_I\right)^2 \ ;$$

$$\overline{\delta}_I = \frac{1}{N}\sum_{t=1}^{N}\delta_{I;t} \text{ ; and}$$

$$\overline{\delta}_M = \frac{1}{N}\sum_{t=1}^{N}\delta_{M;t} \ .$$

Also, after adjusting for bias:

$$\hat{\sigma}_M^2 = \frac{1}{N-2}\left(S_{MM} - \frac{S_{IM}^2}{S_{II}}\right);$$

where:

$$S_{MM} = \sum_{t=1}^{N}\left(\delta_{M;t} - \overline{\delta}_M\right)^2 \ .$$

A.2The regime-switching model

Following Hamilton (1989), the log-likelihood function of the regime-switching model is:

$$l(\theta) = \ln\, f\left(\delta_{M;1} \,|\, \theta\right) + \sum_{t=2}^{N}\ln\, f\left(\delta_{M;t} \,|\, \delta_{M;t-1}, \ldots, \delta_{M;1}, \theta\right);$$

where:

$$\theta = \begin{pmatrix} p_{00} \\ p_{10} \\ g_0 \\ h_0 \\ \sigma_0 \\ g_1 \\ h_1 \\ \sigma_1 \end{pmatrix};$$

$$f\left(\delta_{M;t} \mid \delta_{M;t-1},\ldots,\delta_{M;1},\theta\right) = \begin{cases} f\left(\delta_{M;t}, S_t = 0 \mid \delta_{M;t-1},\ldots,\delta_{M;1},\theta\right) \\ \quad + f\left(\delta_{M;t}, S_t = 1 \mid \delta_{M;t-1},\ldots,\delta_{M;1},\theta\right) \text{ for } t > 1 \\ f\left(\delta_{M;1}, S_1 = 0 \mid \theta\right) + f\left(\delta_{M;1}, S_1 = 1 \mid \theta\right) \text{ for } t = 1; \end{cases}$$

$$f\left(\delta_{M;t}, S_t = s \mid \delta_{M;t-1},\ldots,\delta_{M;1},\theta\right) = \begin{cases} P\left\{S_t = s \mid \delta_{M;t-1},\ldots,\delta_{M;1},\theta\right\} f_s\left(\delta_{M;t} \middle| \theta\right) \text{ for } t > 1; \\ \pi_s f_s\left(\delta_{M;t} \middle| \theta\right) \text{ for } t = 1; \end{cases}$$

$$P\left\{S_t = s \mid \delta_{M;t-1},\ldots,\delta_{M;1},\theta\right\} = p_{0s} P\left\{S_{t-1} = 0 \mid \delta_{M;t-1},\ldots,\delta_{M;1},\theta\right\} + p_{1s} P\left\{S_{t-1} = 1 \mid \delta_{M;t-1},\ldots,\delta_{M;1},\theta\right\};$$

$$\pi_0 = \frac{p_{10}}{p_{10} + p_{01}};$$

$$\pi_1 = \frac{p_{01}}{p_{10} + p_{01}}; \text{ and}$$

$$f_s\left(\delta_{M;t} \middle| \theta\right) = \frac{1}{\sqrt{2\pi}\sigma_s} \exp\left\{-\frac{1}{2\sigma_s^2}\left(\delta_{M;t} - g_s\delta_{I;t} - h_s\right)^2\right\}.$$

A.3The exponential AR model

The log-likelihood function of the exponential AR model is:

$$l(\theta) = -\frac{1}{2}\left\{(N-1)\ln\left(2\pi\hat{\sigma}^2\right) + N - 3\right\};$$

where:

$$\theta = \begin{pmatrix} g \\ h \\ \alpha \end{pmatrix};$$

$$\hat{\sigma}^2 = \frac{1}{N-3}\sum_{t=2}^{N}\left(z_t - he^{\alpha z_{t-1}}\right)^2 \text{ ; and}$$

$$z_t = \delta_{M;t} - g\delta_{I;t}.$$

As explained in section 4.3, the log-likelihood is adjusted to:

$$\tilde{l}(\theta) = \frac{N}{N-1} l(\theta).$$

A.4 The ARCH model

For the ARCH model the log-likelihood function for the estimation of *a* and *b* is:

$$l(\theta) = -\frac{N}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=2}^{N}\left\{\ln(\sigma_t^2) + \left(\frac{z_t}{\sigma_t}\right)^2\right\};$$

where:

$\theta = \begin{pmatrix} a \\ b \end{pmatrix}$;

$z_t = \delta_{M;t} - g\delta_{I;t} - h$; and

$\sigma_t^2 = a + bz_{t-1}^2$.