A STOCHASTIC PROGRAMMING APPROACH TO INTEGRATED ASSET AND LIABILITY MANAGEMENT OF INSURANCE PRODUCTS WITH GUARANTEES

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ABSTRACT

In recent years insurance products have become more complex by providing investors with various guarantees and bonus options. This increase in complexity has provided an impetus for the investigation into integrated asset and liability management frameworks that could realistically address dynamic portfolio allocation in a risk-controlled way.

We propose a multi-stage dynamic stochastic programming model for the integrated asset and liability management of insurance products with guarantees that minimises the down-side risk of these products. We investigate with-profits guarantee funds by including regular bonus payments while keeping the optimisation problem linear.

The uncertainty is represented in terms of arbitrage-free scenario trees using a four-factor term structure model that includes macro economic factors (inflation, capacity utilisation and repo-rates). We construct scenario trees with path dependent intermediate discrete yield curve outcomes suitable for the pricing of fixed income securities. The main focus of the paper is the formulation and implementation of a multi-stage stochastic programming model. The model is back-tested on real market data over a period of five years.

KEYWORDS

Minimum guarantees; asset and liability management; stochastic programming; portfolio optimisation

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1. INTRODUCTION

In recent years multi-stage dynamic stochastic programming models has become a popular tool for integrated asset and liability modelling. In contrast to the usual mean-variance (Markowitz, 1952) approach with a myopic view of managing investment risk over a single period, dynamic stochastic optimisation provides the asset manager with an integrated way to model both assets and liabilities in a flexible manner that takes into account multi-period dynamic asset allocation and the valuation of liabilities under future market conditions. Using this approach the rebalancing of the asset portfolio is modelled explicitly. Examples of the use of dynamic stochastic programming models in asset and liability management can be found in Kouwenberg (2001) and Mulvey, Pauling & Madey (2003). Dempster et al. (2003) show that the dynamic stochastic programming model will automatically hedge the current portfolio allocation against future uncertainties in asset returns and costs of liabilities over the analysis horizon. These models are also flexible enough to take into account multi-period horizons, portfolio constraints such as no short-selling, transaction costs and the investor's level of risk aversion and utility.

In recent years insurance products have become more complex by providing investors with various guarantees and bonus options. This increase in complexity has provided an impetus for the investigation into integrated asset and liability management frameworks that could realistically address dynamic portfolio allocation in a risk-controlled way. Examples of the use of dynamic portfolio optimisation models for asset and liability management in the insurance industry are the Yasuda-Kasai model by Cariño & Ziemba (1998), the Towers Perrin model by Mulvey & Thorlacius (1998) and the CALM model of Consigli & Dempster (1998). More recent contributions specifically in the area of insurance products with minimum guarantees using dynamic stochastic programming as an asset and liability management tool is Dempster et al. (2006) and Consiglio et al. (2006).

Dempster et al. (2006) proposed an asset and liability management framework and gave numerical results for a simple example of a closed-end guaranteed fund where no contributions are allowed after the initial cash outlay. They demonstrated the design of investment products with a guaranteed minimum rate of return focusing on the liability side of the product. Through back-testing they show that the proposed stochastic optimisation framework addresses the risk created by the guarantee in a reasonable way.

Consiglio et al. (2006) study the same type of problem by structuring a portfolio for with-profit guarantee funds in the United Kingdom. The optimisation problem results in a non-linear optimisation problem. They demonstrated how the model can be used to analyse the alternatives to different bonus policies and reserving methods. Consilglio et al. (2001) investigates the asset and liability management of minimum guarantee products for the Italian Industry.

Inspired by the research of Dempster et al. (2006) and Consiglio et al. (2006), we propose a multi-stage dynamic stochastic programming model for the integrated asset and liability management of insurance products with guarantees that minimises the down-side risk of these products. As proposed in Dempster et al. (2006), our model also allows for portfolio rebalancing decisions over a multi-period horizon, as well as for flexible risk management decisions, such as the reinvestment of coupons at intermediate time steps. We investigate with-profits guarantee funds as in Consiglio et al. (2006), by including regular bonus payments. Once these bonuses have been declared, the bonus becomes guaranteed. To keep the optimisation problem linear, we change the way bonuses are declared. We keep the problem linear, for two reasons. The first is that, by keeping the problem linear, we can model the rebalancing of the portfolio at future decision times. By doing so the dynamic stochastic programming model automatically hedge the first stage portfolio allocation against projected future uncertainties in asset returns (see Dempster et al., 2003 and Dempster et al., 2006). The second reason is that the model is flexible enough to take into account portfolio constraints such as the prohibition of short-selling, transaction costs and coupon payments.

We represent the uncertainty in terms of scenario trees by using a four-factor term structure model that includes macro economic factors (inflation, capacity utilisation and repo-rates). We construct scenario trees with path dependent intermediate discrete yield curve outcomes suitable for the pricing of fixed income securities.

In this paper we will discuss the formulation and implementation of the multi-stage stochastic programming model. The model is back-tested on real market data over a period of five years.

2. SCENARIO OPTIMISATION FRAMEWORK

In this section we propose a linear multi-stage dynamic stochastic programming model for the integrated asset and liability management of insurance products with guarantees that minimises the down-side risk of these products.

2.1 MODEL FEATURES

As in Consiglio et al. (2006) we investigate the optimal asset allocation of with-profits guarantee funds, by including regular bonus payments. Once these bonuses have been declared, the bonus becomes guaranteed. We also consider a proprietary company operating a fund on a 90/10 basis, i.e. the policyholder benefits in 90% of the asset share and the share holders 10%. It is assumed that no policyholder contributions are allowed after the initial upfront premium. The time horizon of the fund is T years and the minimum guaranteed rate of return is g on the initial wealth. We use two assets classes,

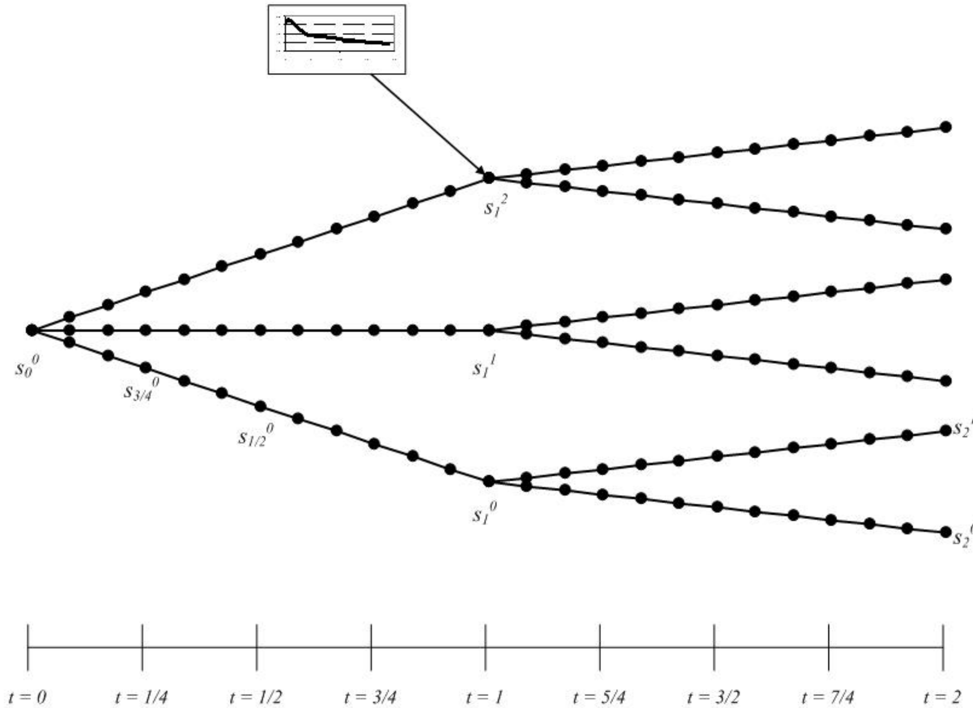

Figure 1. Graphic representation of a scenarios tree

namely, (semi-annual) coupon bearing bonds and equity indices.

We simulate the future yield curves and index movements and construct a scenario tree. A scenario tree is a discrete approximation of the joint distribution of random factors (yield curve and stock indices). We represent the scenario tree in terms of stages (nodes) s_t^v where $t = 0, \frac{1}{12}, \frac{2}{12}, \ldots, 1, \ldots, 2, \ldots, T$ and $v = 1, 2, \ldots, S_t$. The stages at time t are denoted by $\Sigma_t = \left\{ s_t^v \mid v = 1, 2, \ldots, S_t \right\}$. To enforce non-anticipativity, i.e. to prevent foresight of uncertain future events, we order the stages in pairs $\left(s_t^{v(t)}, s_{t+1}^{v(t+1)} \right)$ where the dependence of the index v on t is explicitly indicated. The order of the stages indicates that stage $s_{t+1}^{v(t+1)}$ at time $t+1$ can be reached from stage $s_t^{v(t)}$ at time t. $s_{t+1}^{v(t+1)}$ is the successor stage and $s_t^{v(t)}$ the predecessor stage. That is using the superscript "+" to denote the successor stages, and superscript "-" to denote the predecessors, we have $s_t^{v(t)+} = s_{t+1}^{v(t+1)}$ and $s_{t+1}^{v(t+1)-} = s_t^{v(t)}$. Each stage s_t^v has an associated probability p_t^s such that $\sum_{s \in \Sigma_t} p_t^s = 1$.

Certain times $t_d = 0, 1, 2, \ldots, T-1$ correspond to the annual decision times at which the fund will trade to rebalance its portfolio. We represent the branching of the tree structure with a *tree-string*, which is a string of integers specifying for each decision time

t_d the number of branches for each node in stage Σ_d. This specification gives rise to a balanced scenario tree where each sub tree in the same period has the same number of branches. Figure 1 gives an example of a scenario tree with a (3,2) tree-string, giving a total of 6 scenarios.

2.2 MODEL VARIABLES AND PARAMETERS

The variables and parameters for the model are as follow:

Time sets

$T^{total} = \left\{0, \frac{1}{12}, \frac{2}{12}, \ldots, T\right\}$: set of all times considered in the stochastic program;

$T^d = \{0,1,2,\ldots,T-1\}$: set of decision times;

$T^i = T^{total} \setminus T^d$: set of intermediate times;

$T^c = \left\{\frac{1}{2}, \frac{3}{2}, \ldots, T-\frac{1}{2}\right\}$: set of coupon payment time between decision times;

Note that $T^d \cap T^i = 0$ and $T^c \subset T^i$.

Index sets

$\Sigma_t = \left\{s_t^v \mid v = 1,2,\ldots,S_t\right\}$: set of stages at period t;

SI : set of stock indices;

$B = \{B_\tau\}$: set of government bonds with maturity τ;

$I = SI \bigcup B$: set of all instruments;

Parameters

δ_{B_τ} : coupon rate of a government bond with maturity τ;

F_{B_τ} : face value of a government bond with maturity τ;

$r_{t,\tau}^s$: zero-rate with maturity τ at period t at stage s;

g : minimum guaranteed rate of return;

ρ : regulatory equity to debt ratio;

$r_{t,b}^s$: benchmark rate at period t at stage s;

γ : policyholders' rate of participation in the profits of the firm;

β : target terminal bonus;

$P_{t,i}^{a,s} / P_{t,i}^{b,s}$: ask or bid price of asset $i \in I$ at period t at stage s;

f_a / f_b : proportional transaction costs on ask or bid transactions;

p_t^s : probability of stage s at period t;

Decision variables

$x_t^s = \left\{ x_{t,i}^s \right\}_{i \in I}$: quantities of assets bought at period t at stage s;

$y_t^s = \left\{ y_{t,i}^s \right\}_{i \in I}$: quantities of assets sold at period t at stage s;

$z_t^s = \left\{ z_{t,i}^s \right\}_{i \in I}$: quantities of assets hold at period t at stage s;

A_t^s : value of assets account at period t at stage s;

L_t^s : value of liability account at period t at stage s;

E_t^s : value of equity account at period t at stage s;

c_t^s : amount of equity provided by shareholders at period t at stage s;

SF_t^s : amount of shortfall at period t at stage s;

RB_t^s : regular bonus payment declared at period t at stage s;

TB_T^s : policyholders terminal bonus at period T at stage s;

2.4 BOND PRICING

We assume all bonds to pay semi-annual coupons of δ_{B_τ} and derive bid and ask prices by adding a spread, sp, to the zero-rates. Let $P_{t,B_\tau}^{a,s}$ denote the ask price of a coupon bearing bond with maturity τ at time t:

$$P_{t,B_\tau}^{a,s} = F_{B_\tau} e^{-(\tau - t)\left(r_{t,\tau-t}^s + sp\right)} + \sum_{m=\left\{\frac{\lfloor 2t \rfloor}{2}+\frac{1}{2}, \frac{\lfloor 2t \rfloor}{2}+1, \ldots, \frac{\lfloor 2t \rfloor}{2}+\tau\right\}} \tfrac{1}{2} \delta_{B_\tau} F_{B_\tau} e^{-(m-t)\left(r_{t,(m-t)}^s + sp\right)}, \text{ for}$$

$$t \in T^{total}, \text{ and } s \in \Sigma_t,$$

where the principal amount is discounted in the first term and the coupon payment stream in the second term. Let $P_{t,B_\tau}^{b,s}$ the bid price of the bond with maturity τ at time t:

$$P_{t,B_\tau}^{b,s} = F_{B_\tau} e^{-(\tau - t)\left(r_{t,\tau-t}^s - sp\right)} + \sum_{m=\left\{\frac{\lfloor 2t \rfloor}{2}+\frac{1}{2}, \frac{\lfloor 2t \rfloor}{2}+1, \ldots, \frac{\lfloor 2t \rfloor}{2}+\tau\right\}} \tfrac{1}{2} \delta_{B_\tau} F_{B_\tau} e^{-(m-t)\left(r_{t,(m-t)}^s - sp\right)}, \text{ for}$$

$$t \in T^{total}, \text{ and } s \in \Sigma_t.$$

2.3 VARIABLE DYNAMICS AND CONSTRAINTS

The variable dynamics and constraints for the minimum guarantee problem are:

Cash balance constraints. The cash balance constraints ensure that the amount of cash that is received from selling assets, coupon payments at decision times and equity supplied for shortfall is equal to the amount of assets bought:

$$\sum_{i\in I} P_{0,i}^{a,s} x_{0,i}^{s} \left(1+f_a\right) = A_0^s \text{, for } t\in\{0\} \text{ and } s\in\Sigma_t,$$

$$\sum_{i\in I} P_{t,i}^{b,s} y_{t,i}^{s} \left(1-f_b\right) + \sum_{i\in I\setminus\{SI\}} \tfrac{1}{2}\delta_i F_i y_{t,i}^{s} + c_t^s = \sum_{i\in I} P_{t,i}^{a,s} x_{t,i}^{s} \left(1+f_a\right) \text{, for}$$

$$t\in T^d \setminus\{0\} \text{, and } s\in\Sigma_t.$$

Short sale constraints. The short sale constraints eliminate the possibility of short-selling assets at each stage for each time period:

$$x_{t,i}^s \geq 0 \text{, for all } i\in I,\ t\in T^{total}\setminus\{T\} \text{ and } s\in\Sigma_t,$$

$$y_{t,i}^s \geq 0 \text{, for all } i\in I,\ t\in T^{total}\setminus\{0\} \text{ and } s\in\Sigma_t,$$

$$z_{t,i}^s \geq 0 \text{, for all } i\in I,\ t\in T^{total}\setminus\{T\} \text{ and } s\in\Sigma_t.$$

Inventory constraints. The inventory constraints give the quantity invested in each asset at each stage for each time period:

$$z_{0,i}^s = x_{0,i}^s \text{, for } t\in\{0\} \text{ and } s\in\Sigma_t,$$

$$z_{t,i}^s = z_{t,i}^{s-} + x_{t,i}^s - y_{t,i}^s \text{, for } i\in I,\ t\in T^{total}\setminus\{0\} \text{ and } s\in\Sigma_t.$$

Information constraints. As the portfolio is only rebalanced at decision times, the information constraints ensure that portfolio can not be changed between decision times,

$$x_{t,i}^s = y_{t,i}^s = 0 \text{, for } i\in I,\ t\in T^i\setminus T^c \text{ and } s\in\Sigma_t$$

Coupon reinvestment constraints. The coupon reinvestment constraints ensure that the coupons that are paid at the coupon times are reinvested in the same coupon bearing bonds:

$$x_{t,i}^s = \frac{\tfrac{1}{2}\delta_i F_i z_{t,i}^{s-}}{P_{t,i}^{a,s}\left(1+f_a\right)} \text{, for } i\in I\setminus\{SI\},\ t\in T^c \text{ and } s\in\Sigma_t,$$

$$y_{t,i}^s = 0 \text{, for } i\in I\setminus\{SI\} \text{, for } t\in T^c \text{ and } s\in\Sigma_t,$$

$$x_{t,SI}^s = 0, y_{t,SI}^s = 0 \text{, for } t\in T^c \text{ and } s\in\Sigma_t.$$

Asset account constraints. The asset account constraints determine the value of the asset account at each stage for each time period. The value of the asset account is determined after rebalancing, i.e. any equity c_t^s that has been provided by shareholders to fund shortfalls is taken into account by the cash balance constraints:

$$A_0^s = L_0 + E_0^s \text{, for } t\in\{0\} \text{ and } s\in\Sigma_t,$$

$$A_t^s = \sum_{i\in I} P_{t,i}^{a,s} z_{t,i}^s \left(1+f_a\right) \text{, for } t\in T^{total}\setminus\{T\} \text{ and } s\in\Sigma_t,$$

$$A_T^s = \sum_{i\in I} P_{T,i}^{b,s} z_{T-1/12,i}^{s-} \left(1-f_b\right) + \sum_{i\in I\setminus\{SI\}} \tfrac{1}{2}\delta_i F_i z_{T-1/12,i}^{s-} \text{, for } s\in\Sigma_T.$$

Liability account constraints. The liability account constraints determine the value of the liability account at each stage for each time period. The liability grows at the guaranteed rate of return plus any regular bonus payments that are declared:

$$L_0^s = L_0 \text{, for } t \in \{0\} \text{ and } s \in \Sigma_t \text{,}$$

$$L_t^s = L_{t-\frac{1}{12}}^{s-} e^{\frac{1}{12}g} + RB_t^s \text{, for } t \in T^{total} \setminus \{T\} \text{ and } s \in \Sigma_t \text{.}$$

Equity account constraints. The equity account constraints determine the value of the equity account at each stage for each time period. The equity grows at the one month zero-rate. The shortfall is funded by the shareholders by the infusion of additional equity:

$$E_0^s = c_0^s \text{, for } t \in \{0\} \text{ and } s \in \Sigma_t \text{,}$$

$$E_t^s = E_{t-\frac{1}{12}}^{s-} e^{r_{t-\frac{1}{12},t}^{s-}} + c_t^s \text{, for } t \in T^{total} \setminus \{T\} \text{ and } s \in \Sigma_t \text{.}$$

Regular bonus constraints. The regular bonus constraints determine the amount of the regular bonus payment at each stage for each decision time. To determine the amount of the regular bonus we follow the approach described by Consiglio et al. (2006) which is based on that of Ross (1989) where the regular bonuses are determined by aiming for a target terminal bonus, i.e. the firm wishes the policyholders' terminal benefit to be a fixed portion of the total benefit received. Regular bonuses are assumed to be declared at decision times only (i.e. annually).

It is assumed that the asset account will grow constant at the current benchmark rate, $r_{t,b}^s$, up to termination, giving the terminal asset value as:

$$A_T^s = A_t^{bs} e^{r_{t,b}^s (T-t)} \text{,}$$

where $A_t^{bs} = \sum_{i \in I} P_{t,i}^{b,s} z_{t-\frac{1}{12},i}^{s-} \left(1 - f_b\right) + \sum_{i \in I \setminus \{SI\}} \frac{1}{2} \delta_i F_i z_{t-\frac{1}{12},i}^{s-}$ is the value of the asset account before transactions. It is further assumed that the liabilities will grow at the minimum growth rate, g, up to termination. Furthermore, it is assumed that the regular bonus payment, RB_t^s, that is declared at time t will stay constant through out the remainder of the term and will be invested at the minimum guarantee, g. Thus the terminal liability value is:

$$L_T^s = L_{t-\frac{1}{12}}^{s-} e^{g(T-t+\frac{1}{12})} + RB_t^s \left(\frac{e^{g(T-t)} - 1}{e^g - 1} + e^{g(T-t)} \right) \text{,}$$

where $\left(\frac{e^{g(T-t)} - 1}{e^g - 1} + e^{g(T-t)} \right)$, is the accumulated value of a constant annuity with payment one cash unit from time t to T invested at the minimum guarantee g.

The terminal bonus, $TB_T^s = \gamma\left(A_T^s - L_T^s\right)$, received by the policyholders need to constitute $\beta\%$ of the total amount received by the policyholders:

$$\frac{TB_T^s}{TB_T^s + L_T^s} = \beta .$$

Solving for RB_t^s yields

$$RB_t^s = \frac{\gamma\left(1-\beta\right) A_t^{b,s} e^{r_{t,b}^s (T-t)} - \left(\beta + \gamma\left(1-\beta\right)\right) L_{t-1/12}^{s-} e^{g\left(T-t+1/12\right)}}{\left(\beta + \gamma\left(1-\beta\right)\right)\left(\dfrac{e^{g(T-t)}-1}{e^g - 1} + e^{g(T-t)}\right)} .$$

When the expected terminal asset amount exceeds the expected terminal liability amount regular bonuses will increase. When the expected terminal liability amount exceeds the expected terminal asset amount the regular bonus will be negative. As this will be unfair towards policyholders to declare negative bonuses the constraint is given as follow:

$$RB_t^s >= \frac{\gamma\left(1-\beta\right) A_t^{b,s} e^{r_{t,b}^s (T-t)} - \left(\beta + \gamma\left(1-\beta\right)\right) L_{t-1/12}^{s-} e^{g\left(T-t+1/12\right)}}{\left(\beta + \gamma\left(1-\beta\right)\right)\left(\dfrac{e^{g(T-t)}-1}{e^g - 1} + e^{g(T-t)}\right)} \text{, for}$$

$$t \in \left(T^d \setminus \{0\}\right) \cup \{T\} \text{, and } s \in \Sigma_t ,$$

where $RB_t^s >= 0$ and $RB_t^s = 0$ for $t \in \left(T^i \cup \{0\}\right) \setminus \{T\}$, and $s \in \Sigma_t$. By enforcing the regular bonus constraints the optimisation will determine the regular bonus amount RB_t^s at each decision period.

Consiglio et al. (2006) also consider the *working party approach* based on Chadburn (1997) which is based on work done by Institute of Actuaries Working Party. This approach declares regular bonuses (in return form) to reflect the benchmark return subject to the liability account remaining lower than the value of the *reduced asset account*, where the reduced assets accumulates at 75% of the return on assets. Consiglio et al. (2006) test their model with both these features and find that bonus policies based on aiming for a target terminal bonus outperforms bonus polices based on the working party approach.

Shortfall constraints. The shortfall constraints determine the regulatory shortfall of the portfolio at each stage for each time period. The shortfall is calculated by using the value of the asset account before transaction:

$$SF_0^s + L_0 >= \left(1+\rho\right) L_0^s \text{, for } t \in \{0\} \text{ and } s \in \Sigma_t ,$$

$$SF_t^s + A_t^{bs} >= \left(1+\rho\right) L_{t-1/12}^s e^{g\,1/12} \text{, for}$$

$$t \in \left(T^{total} \setminus \{0\}\right), \text{ and } s \in \Sigma_t$$

where $A_t^{bs} = \sum_{i \in I} P_{t,i}^{b,s} z_{t-1/12,i}^{s-} \left(1 - f_b\right) + \sum_{i \in I \setminus \{SI\}} \frac{1}{2} \delta_i F_i z_{t-1/12,i}^{s-}$ is the value of the asset account before transactions and $SF_t^s >= 0$ for $t \in T^{total}$, and $s \in \Sigma_t$. The shortfall SF_t^s at decision periods are funded by the shareholders equity payment, c_t^s, thus $c_t^s = SF_t^s$ for $t \in \left(T^d\right) \cup \{T\}$ and $s \in \Sigma_t$, and zero at intermediate nodes, $c_t^s = 0$ for $t \in T^i \setminus \{T\}$, and $s \in \Sigma_t$. By enforcing the shortfall constraints the optimisation will determine the amount of equity c_t^s provided by the shareholders at each decision period.

2.4 OBJECTIVE FUNCTION

When managing a minimum guarantee fund there are two main goals to take into account. The first aim is the management of the investment strategies of the fund. The second is to maximise the shareholder value taking into account the minimum guarantee given to policy holders. The shareholders final wealth is given as $\left(1-\gamma\right)\left(\left(A_T - E_T\right) - L_T\right) + E_T$ where $\left(1-\gamma\right)\left(\left(A_T - E_T\right) - L_T\right)$ is the excess amount they receive after the liability and the equity has been paid.

The objective to consider is the maximum expected excess wealth of the shareholders and the minimum average expected shortfall over all periods. Dempster et al. (2006) has shown that examining shortfall at intermediate nodes improves results. The objective function is given as:

$$\max_{\left\{\substack{x_{t,i}^s, y_{t,i}^s, z_{t,i}^s : \\ i \in I, t \in T^d \cup \{T\}, s \in \Sigma_t}\right\}} \left\{ \begin{array}{c} \left(1-\alpha\right) \sum\limits_{s \in \Sigma_T} p^s \left(1-\gamma\right)\left(\left(A_T^s - L_T^s\right) - E_T^s\right) \\ -\alpha \sum\limits_{t \in T^{total}} \sum\limits_{s \in \Sigma_t} p^s \dfrac{SF_t^s}{\left|T^{total}\right|} \end{array} \right\},$$

where the value of $0 \leq \alpha \leq 1$ sets the level of risk aversion and can be chosen freely. If the value of α is to closer 1, more importance is given to the shortfall of the portfolio and less given to the expected excess wealth of the shareholders and hence a more risk-averse portfolio allocation strategy will be taken and visa versa. In the extreme case where $\alpha = 1$ only the shortfall will be minimised and the expected excess wealth will be ignored, and where $\alpha = 0$, the unconstrained case, only maximises the expected excess wealth of the shareholders.

3. RESULTS

In this section we discuss the performance of the model. The first two parts explain the scenario generation algorithm that we use to generate scenario trees which is the input to our mathematical optimisation problem. In the third part we present back-tested results for the model for different levels of the guarantee rate and different levels of risk aversion.

3.1 SCENARIO GENERATION

We estimate the yield curve dynamics with the four-factor yield curve representation of Svensson (1994). The four unobserved factors, level, slope and the two curvature factors, which provide a good representation of the yield curve, are linked to the macro-economic variables by means of a state-space model. We include the following three variables as measures of the state of the economy: manufacturing capacity utilisation, which represents the level of real economic activity relative to potential; the annual percentage change in the inflation index, which represent the inflation rate; and the repo-rate, which represents the monetary policy instrument. According to Diebold et al. (2006) these three macro-economic variables are considered to be the minimum set of fundamentals needed to capture the basic macro-economic dynamics. The model parameters are estimated using a Kalman filter approach. For a complete description of the model and the calibration of the model parameters see Raubenheimer & Kruger (2008).

Raubenheimer & Kruger (2008) also proposes a parallel simulation and clustering approach to create the scenario tree structure as described in Section 2.1. A T-period scenario tree structure is represented as a *tree-string* which is a string of integers specifying for each decision time $t_d \in T^d$ the number of branches (or branching factor) for each node in that time (see Dempster et al., 2006). This gives rise to balanced scenario trees, in which each sub tree in the same period has the same number of branches. Let $\textit{tree-string} = (k_0, k_1, ..., k_d, \ldots, k_{T-1})$ denote a typical tree-string, then the branching factor for decision time t_d, is given by k_d. Figure 1 gives an example of a scenario tree with a (3,2) tree-string, i.e. $k_0 = 3$ and $k_1 = 2$.

The main steps of the algorithm can be outlined as follow:

Step 1: At $t = 0$ create a root node group containing N scenarios. Generate all the scenarios using Monte Carlo simulation and the four-factor yields-macro model. Each scenario is equally likely and consists of T sequential yield curves (in total $T \times N$ yield curves are generated).

Step 2: Set $t := t+1$ and for each group in the previous decision time, calculate the mean scenario and calculate the *relative position* (defined below) of each scenario with respect to the average scenario.

Step 3: For each group, sort the scenarios in descending distance order and group them into k_t equal sized groups.

Step 4: For each new group, find the scenario closest (in absolute value) to its centre, and designate it as the centroid. Assign a probability of $\left(\prod_{i=1}^{s-1} k_i\right)^{-1}$ to each centroid.

Step 5: If $t < T$, go to Step 2, else stop.

As a measure of *relative position* we calculate the "distance" between the discounting factors of the yield curve and that of the average by:

$$D_t^n = \sum_\tau \left(\frac{1}{\left(1+r_{t,\tau}^n\right)^\tau} - \frac{1}{\left(1+r_{t,\tau}^M\right)^\tau} \right),$$

where $r_{t,T}^n$ is the zero-rate with maturity τ and $r_{t,\tau}^M$ the average zero-rate with maturity τ. Note that the relative distance D_t^n can be negative and positive, which means that a yield curve can be positioned to the "left" or to the "right" of the average yield curve. It is necessary to represent each group of scenarios with a single point, which becomes the data in the scenario tree. We use the mean of the group as the notion of the centre.

The scenarios generated are not arbitrage-free (see Klaassen, 2002 and Filipović, 1999). Raubenheimer and Kruger (2008) propose the following method to reduce the arbitrage opportunities:

Step 1: At the root node create a group of N scenarios. Generate all the scenarios using Monte Carlo simulation and the four-factor yields-macro model (as for the scenario tree). Each scenario is equally likely and consists of *T* sequential yield curves.

Step 2: At each decision time of the scenario tree calculate the average of the N generated scenarios (at the root node the current yield curve is used).

Step 3: Then for each average yield curve and the corresponding one-period ahead scenarios solve

$$\frac{1}{N}\sum_{n=1}^{N} e^{-(\tau-1)\left(r_{t+1,\tau-1}^n + c_{t+1,\tau}\right)} = \frac{e^{-\tau r_{t,\tau}}}{e^{-r_{t,1}}}$$

for all maturities, to obtain the yield curve shifts $c_{t+1,\tau}$.

Step 4: Add the amount $c_{t+1,\tau}$ to the original scenario tree yield curves.

Table 1. Tree structure used for back-testing

Year	Tree-string
April 03	5.5.5.5.5 = 3125
April 04	8.8.8.8 = 4096
April 05	15.15.15 = 3375
April 06	56.56 = 3136
April 07	3125

The method removes most of the arbitrage opportunities in the scenario tree with a few opportunities left in sub-trees. For scenario trees with a short horizon all opportunities may be removed. We judge this reduction of arbitrage opportunities as sufficient, since portfolio constraints in the optimisation problem, such as the restriction of short-selling and the inclusion of transaction costs, will eliminate or minimise the effect of the remaining arbitrage opportunities.

3.2 DATA AND INSTRUMENTS

We use six different assets, namely, (semi-annual) coupon bearing bonds with maturities 5, 7, 10, 15 and 19 years and the FTSE/JSE Top 40 equity index. Scenarios for the equity index are generated along with the yield curve by modelling the FTSE\JSE Top 40 index with respect to the three macro-economic variables. We use the Perfect Fit Bond Curves, one of the five BEASSA Zero Coupon Yield Curve series of yield curves (see Bond exchange of South Africa, 2003a), with maturities 1, 2, 3, 6, 9, 12, 15, 18, 21, 24, 36, 48, 60, 72, 84, 96, 108, 120, 132, 144, 156, 168, 180, 192, 204, 216 and 228 months. The curves are derived from government bond data and the technical specifications are described in Bond exchange of South Africa (2003b). We use end-of-month data from August 1999 through to April 2008 and a tree structure with approximately the same number of scenarios. The tree structure used in back-testing is displayed in Table 1.

We use the scenario generation approach to generate the input scenarios for the optimisation problem. The four-factor yields-macro model is fitted to market data up to an initial decision time t and scenario trees are generated from time t to some chosen horizon $t+T$. The optimal first stage/root node decisions are then implemented at time t. The success of the portfolio strategy is measured by its performance with historical data up to time $t+1$. This whole procedure is rolled forward for T trading times. At each decision time t, the parameters of the four-factor yields-macro model are re-estimated using the historical data up to and including time t.

3.3 BACK-TESTED RESULTS

We perform back-tests over a period of five years, from April 2003 through to April 2008, for different levels of minimum guarantee and for different levels of risk-aversion. For each of these back-tests, at different levels, we report the annual expected *excess return on equity* (ExROE), taken to be

$$\sqrt[T]{\sum_{s\in\Sigma_T}\left(\frac{(1-\gamma)\left(A_T^s-L_T^s\right)+\gamma E_T^s}{E_T^s}\right)}-1,$$

at each decision time and the annual actual *excess return on equity*, taken to be

$$\sqrt[T]{\frac{(1-\gamma)\left(A_T-L_T\right)+\gamma E_T}{E_T}}-1.$$

We also report the *expected cost of the guarantee* taken to be the expected present value of the final equity deducting the regulatory equity or equity at the start

$$\sum_{s\in\Sigma_T}\left(\frac{E_T^s}{\prod_{t=1}^{T}\left(1+r_{t,t+1/12}^{s_t^{\nu(t)}}\right)}-E_0\right)p^s$$

where $\left(s_t^{\nu(t)}, s_{t+1/12}^{\nu(t+1/12)}\right)\in\varepsilon$ and the *actual cost of the guarantee*, taken to be

$$\left(\frac{E_T}{\prod_{t=1}^{T}\left(1+r_{t,t+1/12}\right)}-E_0\right).$$

In Figure 2 we present the expected ExROE at decision times and the actual ExROE for different levels of the minimum guarantee. The model over estimates the ExROE, the expected ExROE improves as more data gets available. The actual ExROE decreases as the minimum guarantee increases as would be expected. In Figure 3 we present the expected cost of equity at the decision times and the actual cost of equity. The model firstly over estimates the cost of the guarantee and as more data becomes available the expected cost of the guarantee improves. For lower levels of minimum guarantee the model requires less equity, as the level increases above 9% the amount of equity required increases. Figure 4 presents the performance of the asset account and the liability account at 1%, 9% and 15% minimum guarantee. The asset level stays above the liability level over the entire period. Regular bonuses are paid up to a 15% minimum guarantee. The amounts of regular bonus payments decrease as the level of the guarantee increases.

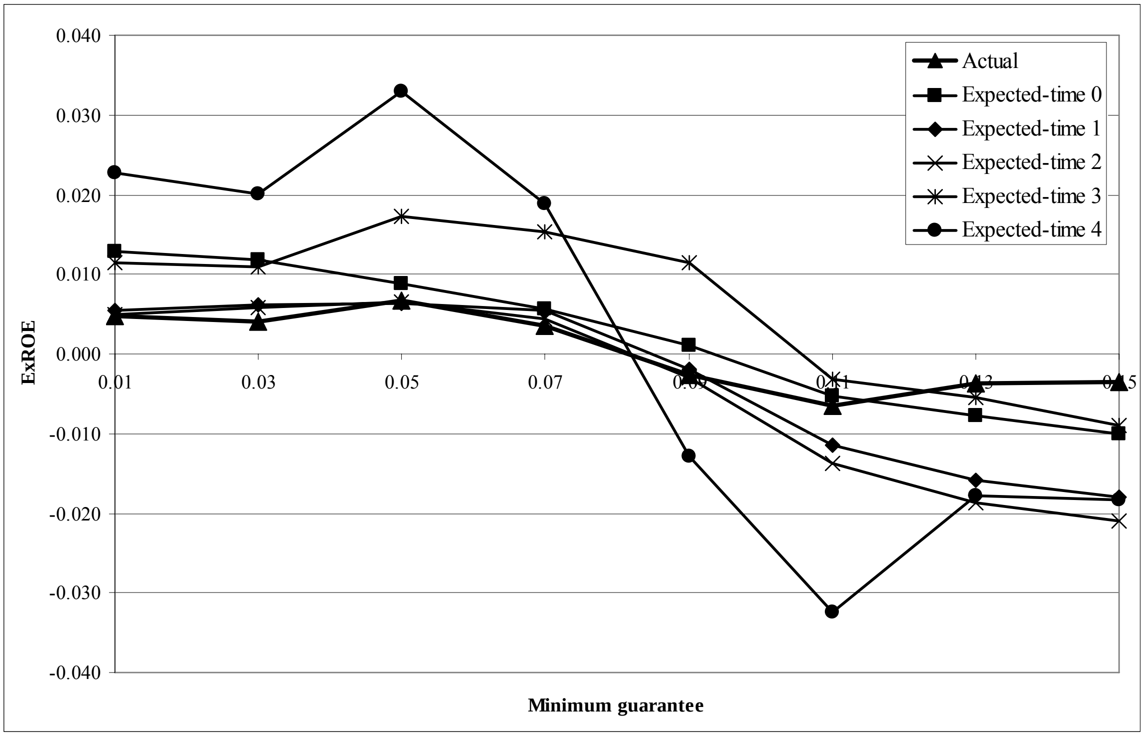

Figure 2. Shareholders annual excess return on equity for different levels of minimum guarantee at $\alpha = 0.5$

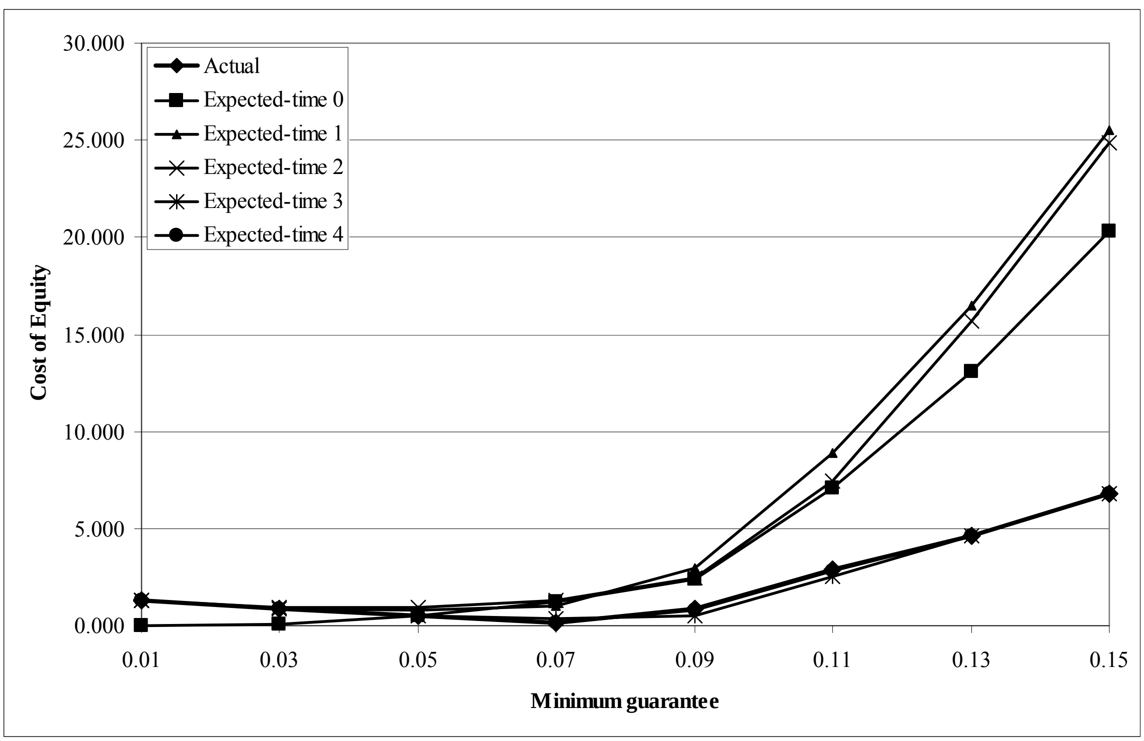

Figure 3. Cost of equity for different levels of minimum guarantee at $\alpha = 0.5$

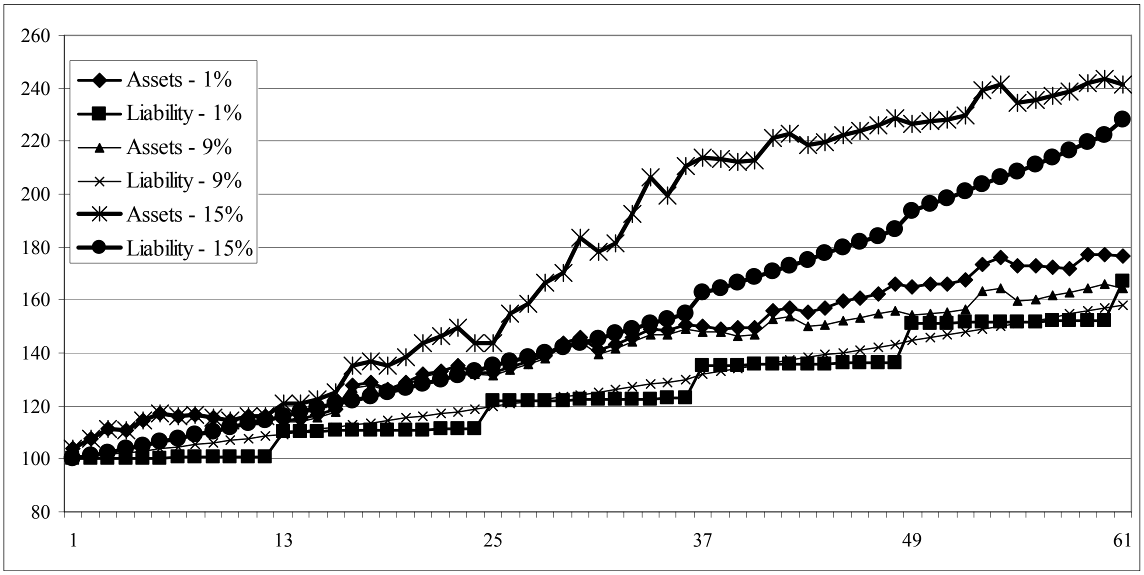

Figure 4. Asset and Liability account at 1%, 9% and 15% minimum guarantee at $\alpha = 0.5$

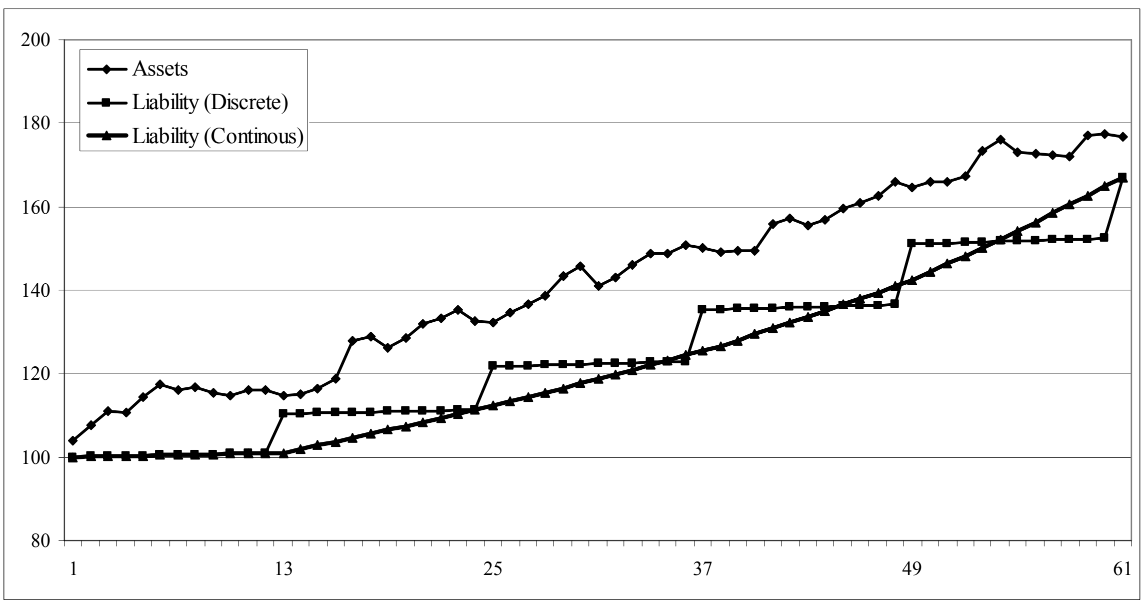

Figure 5. Liabilities with different bonus options at 1% minimum guarantee

Consiglio et al. (2006) specify regular bonuses in return form, which is more realistic than our formulation of discrete annual payments which we define in order to keep the problem linear. Consiglio et al. (2006) assumes that the bonus return, RB_t^s, that is declared at time t will stay constant through out the remainder of the term giving the terminal liability value as:

$$L_T^s = L_t^s e^{g(T-t)} e^{RB_t^s (T-t)} .$$

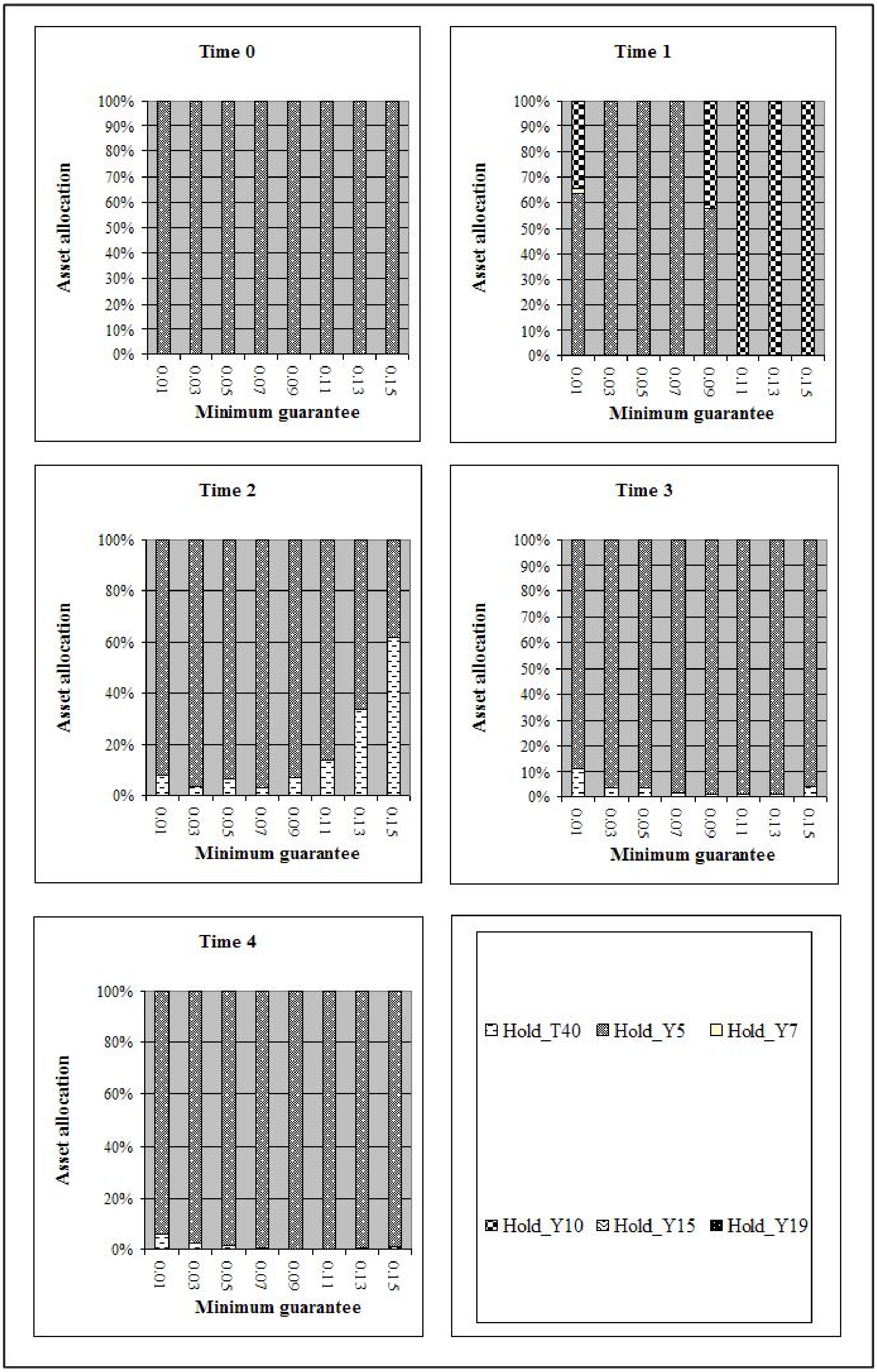

Figure 6. Asset allocation for different levels of minimum guarantee at $\alpha = 0.5$

With all other assumptions staying constant the regular bonus yields:

$$RB_t^s = \max\left[\frac{1}{(T-t)}\ln\left(\frac{\gamma(1-\beta)A_t^{b,s}e^{r_{t,b}^s(T-t)}}{(\beta+\gamma(1-\beta))L_t^s e^{g(T-t)}}\right),0\right].$$

With back-testing performance we have also implemented the liability process proposed by Consiglio et al. (2006). Figure 5 shows that our discrete approximation of bonuses mimics the more realistic approach of Consiglio et al. (2006). Recall that our approach has the added advantage of keeping the overall problem linear which allows us to include more realistic portfolio management constraints.

Figure 6 shows the first stage optimal asset allocation at decision times for different levels of the minimum guarantee. The time 0 asset allocation does not seem consistent with the rest of the asset allocations and is likely due to the short time series available for parameter estimation, but as time progresses and more data becomes available the allocation improves. At reasonable levels of minimum guarantee the portfolio is less aggressive and allocates less in the risky asset. At low levels of the minimum guarantee the asset account tends to be less aggressive at the begging of the term and more aggressive at the end, for higher levels the asset account tends to be more aggressive at the begging of the term.

In Figure 7 we present the expected ExROE at decision times and the actual ExROE for different levels of risk-aversion at a 9% minimum guarantee. The model again over estimates the ExROE, the expected ExROE improves as more data becomes available. The ExROE decreases as the level of risk-aversion increases, as would be expected. The ExROE steadily decreases as the risk-aversion level moves from 0 to 1. In Figure 8 we present the expected cost of equity at the decision times and the actual cost of equity. Again the model firstly over estimates the cost of guarantee and as more data becomes available the expected cost of guarantee improves. The expected cost of equity decreases as the as the level of risk-aversion increases up to 0.4 where it is possible to achieve a positive ExROE. After 0.4 the cost of equity increases when a negative ExROE is achieved. Figure 9 presents the performance of the asset account and the liability account at 0 and 1 level of risk-aversion at 9% minimum guarantee. The asset level stays above the liability level over the entire period. At 0 level of risk-aversion the model tends to be more aggressive at 1 level of risk-aversion the model is more conservative.

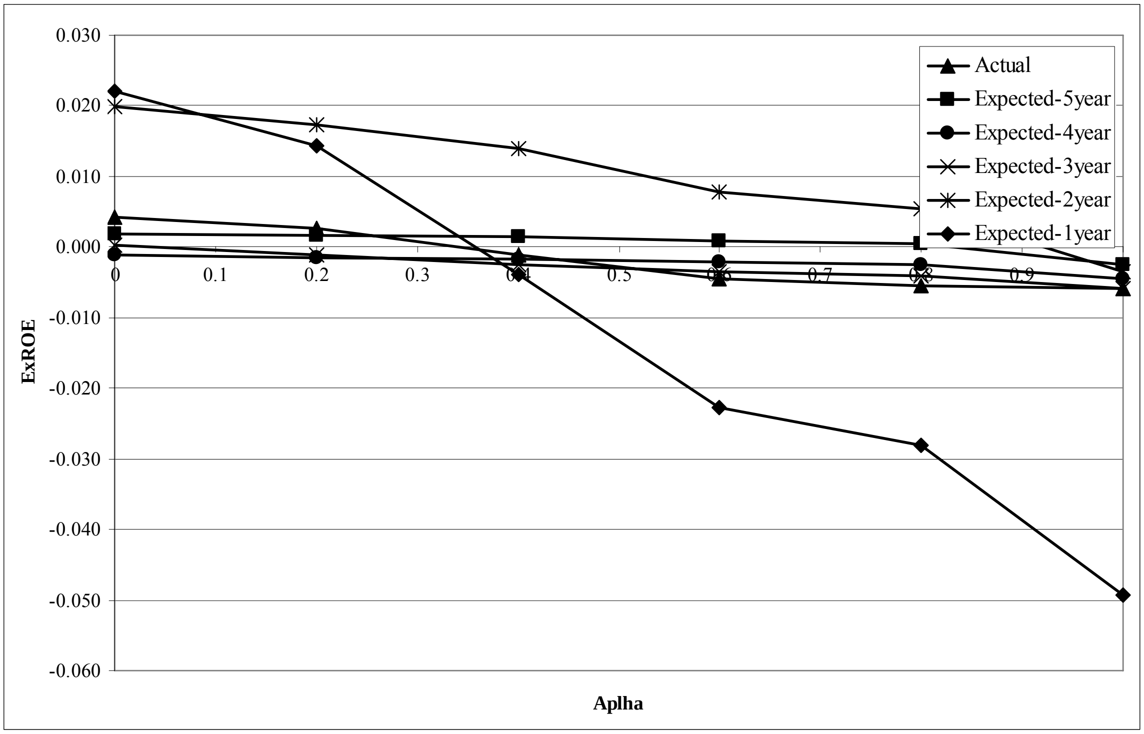

Figure 7. Shareholders annual excess return on equity for different levels of risk-aversion at 9% minimum guarantee

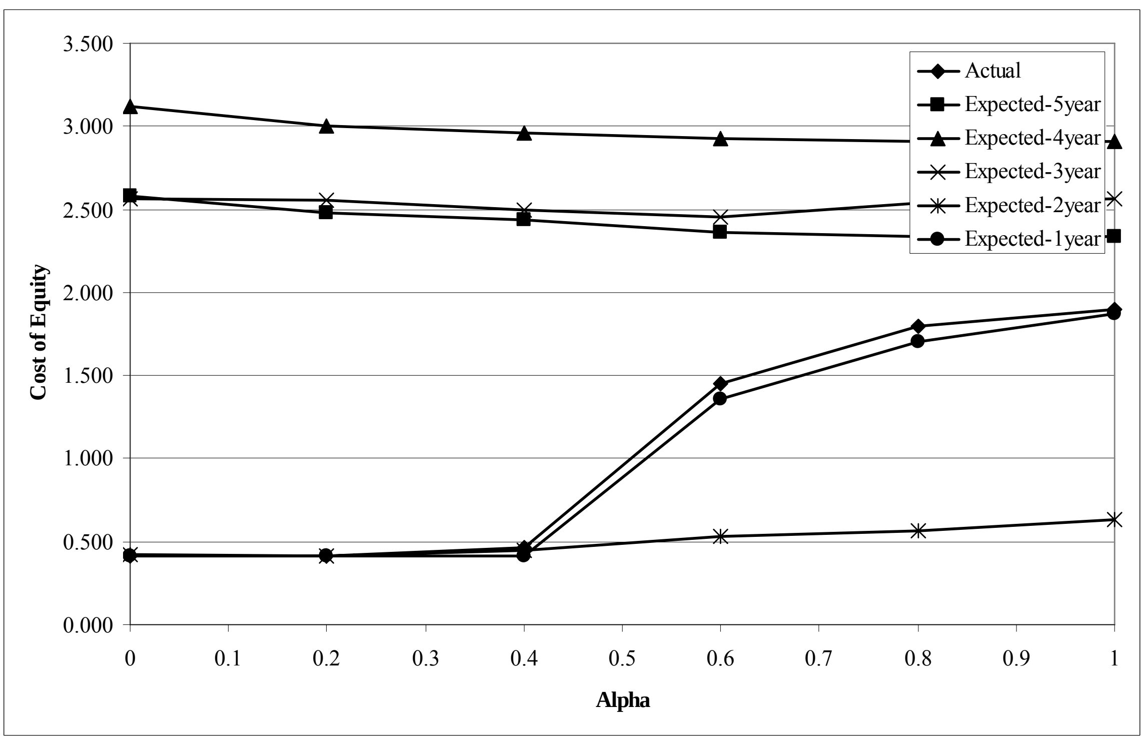

Figure 8. Cost of equity for different levels of risk-aversion at 9% minimum guarantee

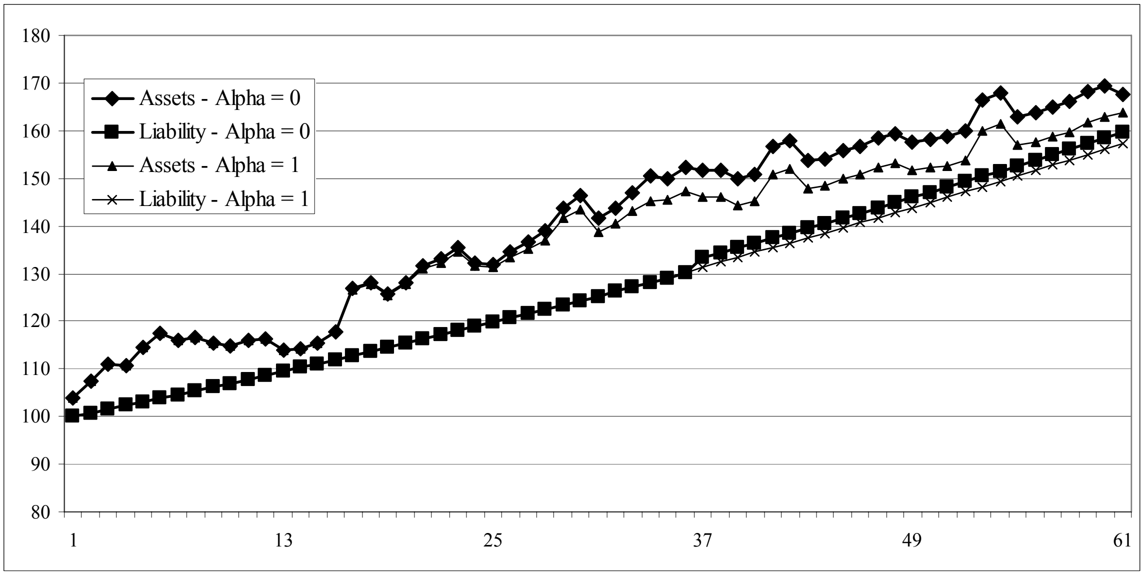

Figure 9. Asset and Liability account at 0 and 1 level of risk-aversion at 9% minimum guarantee

Figure 10 shows the first stage optimal asset allocation at decision times for different levels of risk-aversion for at 9% minimum guarantee. The time 0 asset allocation again does not seem consistent with the rest of the asset allocations and, as mentioned previously, is likely due to the short time series available for parameter estimation. As the level of risk-aversion increases the portfolio is more conservative and allocates less in the risky asset as in lower levels of risk aversion.

4. CONCLUSION

In this paper we have presented a multi-stage dynamic stochastic programming model for the integrated asset and liability management of insurance products with guarantees that minimises the down-side risk of these products. We included regular bonus payments and kept the optimisation problem linear, which enables us to model the rebalancing of the portfolio at future decision times. Also, by keeping the optimisation problem linear, the model is flexible enough to take into account portfolio constraints such as the prohibition of short-selling, transaction costs and coupon payments. We have also shown that our bonus assumption mimics those proposed by Consiglio et al. (2006). We have shown the model features at different levels of minimum guarantee and different levels of risk aversion. As Consiglio et al. (2006) have shown that the model can also be used for analysing the investment decision made by the insurance firm. Future extensions may look at the inclusion of other policy features.

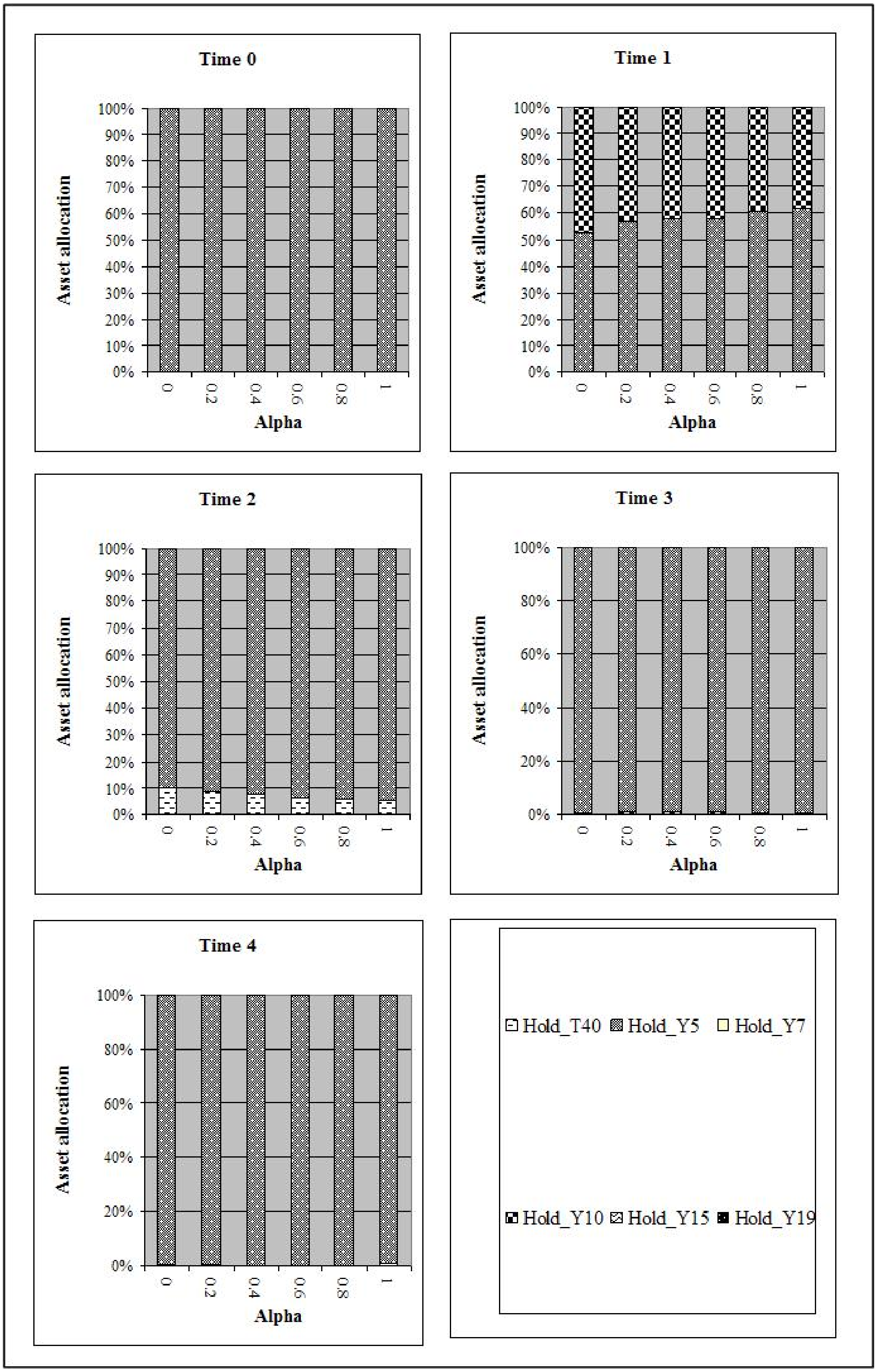

Figure 10. Asset allocation different levels of risk-aversion at 9% minimum guarantee

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