The quantification of type-2 prudence in asset allocation by the trustees of a retirement fund

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ABSTRACT

In this paper, consideration is given to the normative use of expected-utility theory for the purposes of asset allocation by the trustees of retirement funds. Both defined-contribution (DC) and defined-benefit (DB) funds are considered. In the case of a DC fund the trustees' asset-allocation problem relates to a fund in which there is no investment-channel choice or to a fund in which there is a default investment channel or investment strategy. In the DB case the problem relates to the assets of the fund as a whole. Because the application of expected-utility theory envisaged in this paper is normative, many of the problems with expected utility as a descriptive theory are not relevant. Nevertheless, relevant literature on behavioural finance is discussed.

In the actuarial literature, applications of expected-utility theory to retirement-fund decision-making involve a plethora of different functional forms of the utility function, and of different definitions of the argument (i.e. the input variable) of the utility function. Levels of risk aversion vary widely. Time intervals and time horizons also vary. Some authors use dynamic programming, others do not.

In order to inform the choices that need to be made by actuaries, the authors critically discuss the relative merits of these alternatives. In the paper, a distinction is drawn between 'type-1 prudence', which relates to deliberate conservatism on the part of actuaries in the setting of assumptions and the determination of model parameters, and 'type-2 prudence', which relates to the risk aversion of the trustees. The intention of the research was to quantify type-2 prudence for the purposes of asset allocation. It is argued that the requirement of type-2 prudence implies that certain criteria should be satisfied by a utility function for normative use by the trustees of a retirement fund.

A new class of utility functions, referred to as the 'weighted average relative risk aversion' (WARRA) class, which satisfies the required criteria, is introduced. Besides satisfying the required criteria, the WARRA class has some advantages, which are explained in the paper. Methods of determining the parameters of a utility function of the proposed class are described. In particular, it is shown how, for a particular fund, the choice of a utility function may be informed by the utility functions elicited from a sample of members of the fund.

Counter-intuitive results found by other authors paradoxically suggest that the trustees of a retirement fund should invest more in risky assets when the fund is in shortfall and less when it is in surplus. Illustrative results from the application of the method proposed in this paper resolve that paradox. The sensitivity of asset allocations to the assumptions made is analysed.

KEYWORDS

Expected utility; HARA class; weighted average relative risk aversion; WARRA class; retirement funds; trustees; type-2 prudence

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1. INTRODUCTION

1.1 Overview

In this paper, consideration is given to the normative use of expected utility (EU) theory for the purposes of asset allocation by the trustees of retirement funds. Normative applications relate to the use of EU theory in situations in which decision-makers have obligations, or in which they wish to apply reasonable axioms to their decision-making. Descriptive applications relate to the use of that theory in describing how decision-makers make decisions. General literature on this subject is reviewed in section 2, but literature of particular relevance to the issues here addressed is deferred to section 3. Because the application of EU theory envisaged in this paper is normative, many of the problems with expected utility as a descriptive theory are not relevant. Relevant literature on behavioural finance is, however, discussed in section 2.

In the actuarial literature, applications of EU theory to retirement-fund decision-making involve a plethora of different functional forms of utility functions, and of different definitions of the argument of the utility function. Levels of risk aversion vary widely. Time intervals and time horizons also vary. Some authors use dynamic programming, others do not. In section 3, in order to inform the choices that need to be made by actuaries, the authors critically discuss the relative merits of these alternatives.

In section 4 a new class of utility functions that satisfies the required criteria is introduced. Methods of determining the parameters of a utility function of the

proposed class are described. In particular, it is shown how, for a particular fund, the choice of a utility function may be informed by the utility functions elicited from a sample of members of the fund.

Counter-intuitive results found by other authors paradoxically suggest that the trustees of a retirement fund should invest more in risky assets when the fund is in shortfall and less when it is in surplus. In section 5 this literature is reviewed. Also in that section, illustrative results from the application of the method proposed in this paper are presented. Those results resolve that paradox. They also illustrate the application of EU theory to asset allocation using discrete intervals, with future liability cashflows and a model of risky assets defined by the user. The sensitivity of asset allocations to the assumptions made is analysed.

The findings are summarised in section 6, together with suggestions for further research.

1.2 Terminology

As discussed in section 2, the legal provisions relating to the control of retirement funds vary between jurisdictions, but the members of the controlling body of a retirement fund are in general fiduciaries of the moneys entrusted to them for the prospective beneficiaries of the fund. In this paper they are referred to as 'trustees'. As such, though they may in some countries be referred to collectively as a 'board', they are jointly and severally accountable not as representatives of constituencies of stakeholders but as fiduciaries for the moneys entrusted to them. This requirement raises implicit questions as to the criteria to be used in order to establish whether a particular decision meets it. For the purposes of such decisions a normative approach to decision-making is required—in particular a normative theory that quantifies the requirements not only of risk aversion but also of prudence.

'Prudence' may refer to the assumptions made by actuaries in the valuation of pension schemes (i.e. to deliberately conservative or 'somewhat pessimistic' assumptions—Haberman et al., 2003), or it may refer to the risk appetite of the trustees as decision-makers. More attention is given in the actuarial literature to the former than to the latter. For example, Thornton & Wilson (1992: 275) define 'prudence' as a probability of 60% that events will prove more favourable in the long term. This compares with a 'best estimate' and a 'cautious' basis, for which there are probabilities of 50% and 70% that events will prove more favourable in the long term. These suggestions are criticised by Cowling, Gordon & Speed (2005: 72–3).

In this paper a distinction is therefore drawn between 'type-1 prudence', which relates to deliberate conservatism in the setting of assumptions and the determination of model parameters as contemplated in the preceding paragraph, and 'type-2 prudence', which relates to the risk aversion of the trustees. In terms of expected-utility theory, type-1 prudence relates to the distribution of the outcome (i.e. the argument of the utility function) and type-2 prudence to the specification of the utility function in terms of that outcome.

To clarify this distinction we may consider the expression:

$$E = \int_0^\infty u(x) f(x) dx \,;$$

where:

$u(x)$ is the decision-maker's utility of wealth x at a time horizon; and

$f(\bullet)$ is the probability density function of her/his wealth at that time horizon.

Under EU theory the decision-maker's objective is to maximise E. Here type-1 prudence focuses on $f(\bullet)$. For example, it would involve setting a deliberately low value to the expected value of the decision-maker's wealth at the time horizon and a deliberately high value to its standard deviation. Type-2 prudence, on the other hand, focuses on the utility function $u(x)$. For example, it would involve adopting a utility function with high risk aversion.

The problem with type-1 prudence is that it fails to quantify the level of prudence required. If every assumption on which the distribution of the outcome is based is set at a deliberately prudent level, the result is likely to be excessively prudent. It may also be argued that it is more rational to apply prudence to that part of the decision function which is explicitly designed to account for the level of prudence required than to make the distribution arbitrarily unrealistic. Provided it is possible to establish a utility function that, on the one hand, is not arbitrary and on the other reflects the required level of prudence, the proper place for the application of prudence is therefore in the specification of the utility function. The intention of the research was to quantify type-2 prudence for the purposes of asset allocation by the trustees of a retirement fund. It is argued that the requirement of type-2 prudence implies that certain criteria should be satisfied by a utility function for normative use by the trustees of a retirement fund. Whilst the elicitation of utility functions inevitably involves subjective choices, it is also demonstrated in this paper how an arbitrary choice of utility function may be avoided.

The requirement of type-2 prudence is understood to imply risk aversion in the sense that the trustees require extra returns for extra risks and the quantification of further requirements of prudence is discussed in section 3.7.

Reference is made in this paper to 'retirement funds'. Following usage in certain countries, the intention behind the use of this expression is to include both 'pension funds' (which pay pensions after retirement) and 'provident funds' (which pay lump sums on retirement). This includes both defined-benefit (DB) and defined-contribution (DC) retirement funds.

2. GENERAL REVIEW OF LITERATURE AND DOCUMENTATION

In this section relevant literature and legislative and regulatory documentation is reviewed. The intention of this review is to explore the requirement of prudence as it relates to the trustees of retirement funds, to consider the contribution that behavioural finance has made to the theory of decision-making in the context of retirement funds,

to defend the normative validity EU theory for decision-making by a trustee and to focus on its relevance to asset-allocation decisions in retirement funds.

Whilst some criticism of the literature by other authors is presented here, criticism of the literature relating explicitly to the issues discussed in section 3 is deferred to that section.

2.1 The Requirement of Prudence

In the European Union, a "Directive on the activities and supervision of institutions for occupational retirement provision" (the IORP Directive)[1] establishes a 'prudent person rule' as the 'underlying principle for … investment' by occupational pension schemes'.[2] In terms of the 'investment rules' laid down in that document,[3] the prudent person rule is spelt out in some detail. Those provisions set out what the trustees of an occupational pension scheme must do in order to satisfy this rule. In particular, clauses (1)(a) and (b) require that:

> (a) the assets shall be invested in the best interests of members and beneficiaries…;
> (b) the assets shall be invested in such a manner as to ensure the security, quality, liquidity and profitability of the portfolio as a whole.…

There is no explicit statement as to how trustees are to offset risk against expected return. For example, one would in the first instance expect that trustees should not be risk-seeking. Such implications are presumably left to interpretation of 'prudence' in its broader sense.

In terms of UK legislation,[4] the trustees of occupational pension schemes in that country must maintain a statement of investment principles, which must cover, *inter alia*, their policy about risk and expected return.[5] Here again, though, there is no explicit statement that relates the comparison of risk and expected return to the requirement of prudence.

In the USA, in terms of regulations issued by the Department of Labor under the Employment Retirement Income Security Act 1974 (ERISA),[6] a pension-fund trustee must

> discharge his [*sic*] duties with respect to a plan … with the … prudence … that a prudent man acting in a like capacity and familiar with such matters would use in the conduct of an enterprise of a like character and with like aims.

1 Directive 2003/41/EC of the European Parliament and of the Council of 3 June 2003 on the activities and supervision of institutions for occupational retirement provision
2 ibid.: preamble, clause 6
3 ibid.: article 18 (1)
4 Pensions Act, 1995 c. 26
5 ibid.: section 35
6 United States Code: Title 29: Chapter 18: Employee Retirement Income Security Program: §1104 – Fiduciary duties, www.law.cornell.edu/uscode/text/29/chapter-18

Again there is no explicit statement that relates the comparison of risk and expected return to the requirement of prudence. However, as stated by Watchman (2011):

> From a reading of the US, English and Scots case law and a review of legislation and regulatory guidance on fiduciary duties and pension fund trustees, a core of common law fiduciary duties applicable to a pension fund trustee … may be suggested to include the duties of loyalty, good faith, protection of the pension fund's beneficiaries' (both present and future) best interests, fairness, *prudence* and acting for a proper purpose. (emphasis added)

According to the Legal Information Institute, the prudent investor rule in the United States "has changed over time to reflect the Modern Portfolio Theory." This suggests that prudence should be measured in terms of optimisation in mean–variance space; it does not necessarily imply the application of EU theory.[7]

In the United States, since the move from DB to DC, section 401(k) accounts have become increasingly widely used as instruments of saving for retirement. Employees may contribute to such an account on a tax-deferred basis and may have options as to how their accumulated funds are invested. A portion of the employee's contribution may be matched by the employer. These plans are now the most popular retirement-savings vehicle in the USA. (Poterba, Venti & Wise, 2010) The focus of research on decision-making for retirement has therefore tended to be on 401(k) plans.

Because a 401(k) plan is a DC plan, employees carry the risk of poor investment performance. If an employer establishes a 401(k) plan for its employees, it generally offers options with regard to the investment of their accumulated accounts. To avoid problems regarding the failure of employees to make such a choice, and either explicitly or implicitly to indicate the employer's suggested choice, a default option is generally applied. To avoid problems with employees approaching retirement, and because of concerns that it might be sued for having given a bad suggestion, an employer may appoint a company to provide advisory services to its employees (Scott & Stein, 2004). However, the onus still rests on the employer to exercise prudence in the selection and monitoring of such an adviser. (ibid.) This in turn suggests that a company so appointed must exercise prudence in the provision of advice. Statman (2004) reports that "people have begun suing advisers and employers who let them concentrate their portfolio in stocks." If, given the information available at the time of the advice, it may be shown that the advice was prudent, advisers and employers would be better placed to defend themselves against such claims.

In the UK the 'personal pension scheme' fulfils a similar role to that of the 401(k) plan, but they have not become as popular.[8]

In South Africa, similarly to the UK, the board of trustees of an occupational retirement fund must maintain an investment policy statement, which must, *inter alia*,

7 www.law.cornell.edu/wex/Prudent_Investor_Rule
8 Office for National Statistics. Pension Trends: Chapter 8: Pension contributions, 17 July 2012

> [describe] a fund's general investment philosophy and objectives as determined by its liability profile and risk appetite.[9]

The prudent-person principle implicitly applies.[10] In that country most occupational retirement funds are now DC funds, and most of those funds provide investment choice, with default options selected by the trustees. For those funds the trustees must apply the prudent-person principle to the provision of investment channels and the selection of a default channel.

In Australia, a 'registrable superannuation entity licensee' must similarly establish and maintain

> a board-approved risk appetite statement and a designated risk management function (or role) that has responsibility for assisting the development of the risk management framework.[11]

Again, in South Africa and in Australia, there is no explicit statement that relates the comparison of risk and expected return to the requirement of prudence. Again, though, fiduciary duties are well established at common law and include the requirement of prudence.

In summary, whilst the legislative and regulatory requirements of the countries considered above are not explicit about the risk appetite of trustees of pension schemes, they do explicitly or implicitly require fiduciary duties including prudence, and common law in this regard to be well established. Part of the reason for the vagueness of the law with regard to risk appetite is the absence of research on the relationship between prudence and risk appetite. It is the task of this paper to address that relationship.

2.2 Behavioural Finance

In order to address the relationship between prudence (specifically type-2 prudence) and risk appetite we need to do so in the framework of a theory of decision-making under uncertainty.

Since Von Neumann & Morgenstern (1947) suggested the use of EU theory to explain decisions made by rational agents under uncertainty, it has been found that that theory fails to explain such decisions. As stated by Barberis & Thaler (2003):

> … it has become clear that basic facts about the aggregate stock market, the cross-section of average returns and individual trading behavior are not easily understood in this framework.

9 Regulation 28 substituted by GN R183/2011 in terms of section 26 of the Pension Funds Act, no. 24, 1956
10 ibid.
11 Prudential Standard SPS 220: Risk Management

As Shleifer (2000) put it:

> … individuals do not assess risky gambles following the precepts of von Neumann–Morgenstern rationality.

In the light of these findings, behavioural economists are quite justified in rejecting EU theory as a descriptive theory. In response, the field of behavioural finance has explored alternative theories to explain how agents make decisions under uncertainty. As Barberis & Thaler (op. cit.) point out:

> … experimental work in the decades after [Von Neumann & Morgenstern (op. cit.)] has shown that people systematically violate EU theory when choosing among risky gambles. In response to this, there has been an explosion of work on so-called non-EU theories, all of them trying to do a better job of matching the experimental evidence. Some of the better known models include weighted utility theory, implicit EU; disappointment aversion; regret theory, rank-dependent utility theories, and prospect theory.[12]

Prospect theory (Kahneman & Tvesky, 1979) has been particularly important in explaining decision-making under uncertainty. It may be compared with EU theory as follows. EU theory requires the maximisation of:

$$E = \int_0^\infty u(x) f(x) dx \,;$$

where:

$u(x)$ is the decision-maker's utility of wealth x at a time horizon; and
$f(\bullet)$ is the probability density function of her/his wealth at that time horizon.

Prospect theory proposes the maximisation of:

$$P = \int_{-\infty}^{\infty} v(x) g\big(f(x)\big) dx \,;$$

where:

$v(x)$ is the value to the decision-maker of a gain of x relative to her/his current wealth at the time horizon; and
$g(\bullet)$ is a transformation of the probability density function representing the weighting applied by the decision-maker to a given probability.

Figure 1 shows a typical plot of the value function $v(x)$ and Figure 2 shows a typical plot of the weighting function $g(\bullet)$.

From Figure 1 it may be seen that a typical value function shows an aspiration level at the current level of wealth. This means that, over the range of losses, the

12 cf. Barberis & Thaler (op. cit.) for sources

decision-maker is risk-seeking: she/he deliberately takes risks in order to avoid losses. As discussed below, this is imprudent.

From Figure 2 it may be seen that decision-makers' subjective probabilities are greater than the true probabilities at low levels and lower at medium to high levels. For objectively determined probabilities it would be irrational for decision-makers to weight their probabilities in this way. Again, such behaviour is imprudent.

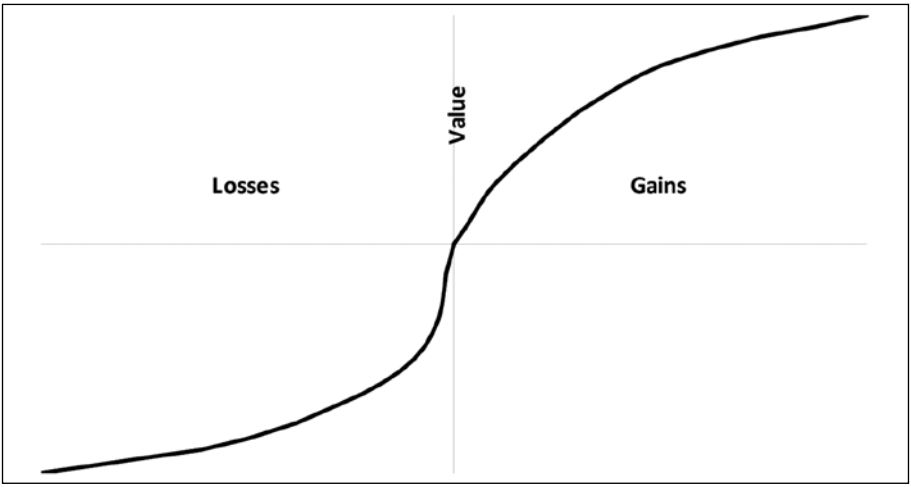

Figure 1 A typical value function

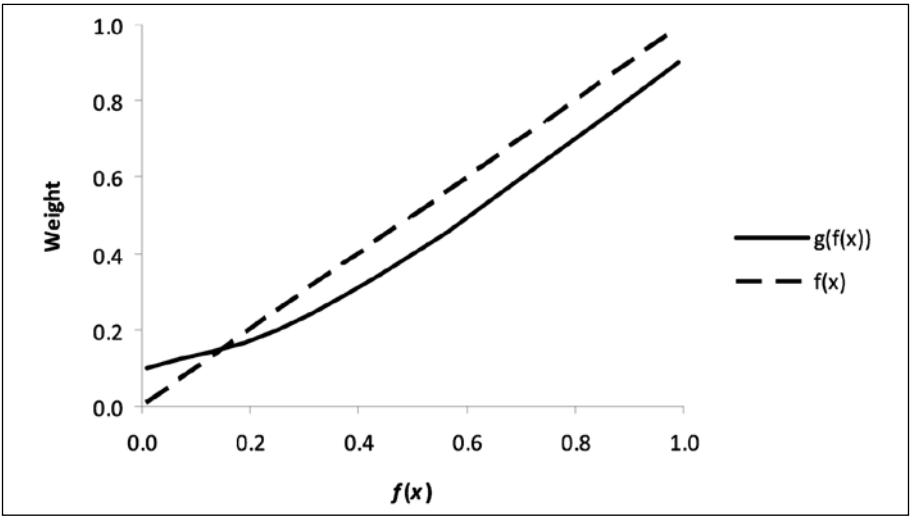

Figure 2 A typical weighting function

Thus, while prospect theory better explains how decision-makers do make decisions, it fails to explain how prudent decision-makers should make decisions. For the former purpose we need a good descriptive theory, whereas for the latter we need a good normative theory. As pointed out by Barberis & Thaler (op. cit.):

> … prospect theory has no aspirations as a normative theory; it simply tries to capture people's attitudes to risky gambles as parsimoniously as possible. Indeed, Tversky & Kahnemann (1986) argue convincingly that normative approaches are doomed to failure, because people routinely make choices that are simply impossible to justify on normative grounds …

Tversky & Kahneman (op. cit.) do not actually argue that normative approaches are "doomed to failure." These are Barberis & Thaler's (op. cit.) words. In fact Tversky & Kahneman (op. cit.) state that:

> Perhaps the major finding of the present article is that the axioms of rational choice are generally satisfied in transparent situations and often violated in nontransparent ones. For example, when the relation of stochastic dominance is transparent …, practically everyone selects the dominant prospect. However, when these problems are framed so that the relation of dominance is no longer transparent …, most respondents violate dominance, as predicted.
>
> The … normative and the descriptive analyses of choice, [they state,] should be viewed as separate enterprises.

It is clear from statements like that of Barberis & Thaler (op. cit.) that there has been a paradigm shift in the economics literature from normative theories to descriptive theories. Economists are clearly not interested in theories that do not describe how agents make decisions under uncertainty. It should be recognised, though, that normative approaches are not doomed to failure as normative theories, but as descriptive theories. It is the task of a descriptive decision theory to explain how agents actually make decisions. A good descriptive theory must therefore use hypothetico-deductive methods based on empirical evidence. On the other hand it is the task of a normative theory to state what decision should be made if the decision-maker accepts the axioms on which it is based. A good normative theory must therefore be based on axioms that are acceptable in principle to the decision-maker (though they may be flouted in practice). Where, as in the example of Tversky & Kahneman (op. cit.), irrationality is merely the effect of a lack of transparency, it is the task of the adviser to create the transparency required for a rational decision.

"In broad terms," as Barberis & Thaler (op. cit.) state, "[behavioral finance] argues that some financial phenomena can be better understood using models in which some agents are *not* fully rational." If a prudent person wishes to make a rational decision, she/he should not use a descriptive theory, but a normative theory. In giving advice to trustees, actuaries need a normative theory.

This does not mean that behavioural finance has no relevance to normative decision-making by fiduciaries. Probability weighting (Kahneman & Tversky, op. cit.), Barberis & Thaler's (op. cit.) finding that people routinely make choices that are simply impossible to justify on normative grounds and the violation of rational choice in non-transparent situations all imply that employees make irrational decisions. Findings such as that of Kahneman & Tversky that value functions show aspiration levels imply that employees make risk-seeking decisions. Indeed, it is because of the irrationality and risk-seeking behaviour of employees that prudent fiduciaries are required. Such fiduciaries must therefore be aware of the biases that may result from the irrationality and risk-seeking behaviour of employees. For example:

1) "Automatic enrollment (in which newly eligible employees are enrolled at a default savings rate and in a default investment option) raises participation rates dramatically." (Madrian & Shea, 2001)
2) Under the "Save More Tomorrow" plan, by Thaler & Benartzi (2004), "workers agree to have their plan contributions increased regularly in the future… They found that workers in one firm were more willing to use this feature than to agree to a one-time increase in saving rate recommended by a financial planner. Over time, these workers ended up saving more than the planner had originally recommended." (MacFarland, Marconi & Utkus, 2004)
3) "Because not all participants are interested in making active and well-informed choices, there may be a greater role for negative elections, default choices, investment advice, and managed 401(k) accounts." (ibid.)
4) "… new approaches to financial education [are required], with greater emphasis on simpler decisions, less information, reduced complexity, and fewer choices." (ibid.)

Similarly, Saliterman & Sheckly (2004) observe that:

> Many of the economic models … simply 'describe investors as they are' implying that these models cannot explain how to change investors' approaches to decisionmaking (Statman, op. cit.). … To resolve the retirement savings problem, the approach of accepting 401(k) participants 'as they are'… may have to be supplemented with tactics for changing how participants think about their retirement plans. Educational programs designed for this purpose, however, have a poor track record.

As observed in section 2, the focus of research on decision-making for retirement in the USA has tended to be on 401(k) plans and in particular on the provision by employers, and the exercise by employees, of investment options. The above examples presuppose that high participation rates, high contribution rates, active and well-informed choices and appropriate financial education are desirable. Whilst such presuppositions appear to be reasonable, what they do not address is how to determine prudent advice and education for investment choice, and in particular the relationship between type-2 prudence and risk appetite.

It is clear from the above discussion that employees fail to make rational decisions. However, not only do employees fail to make rational decisions, Clark, Caerlewy-Smith and Marshall (2006) find that trustees themselves have ill-considered discount functions and risk-aversion and are poor probability analysts and inefficient users of information. The application of normative decision theory together with well-informed stochastic models of the variables involved, would offer a solution to such problems. As stated by Shiller (2003):

> … both approaches to finance, the behavioral approach, and the rational optimising approach, have their own contributions to make …

2.3 The Normative Validity of EU Theory for Decision-making by a Trustee

The shortcomings of EU theory as a descriptive theory are well documented and have been discussed by one of the authors (Thomson, 2003a). As explained in that paper, and as further explained in section 2.2 above, these shortcomings do not affect EU theory as a normative theory. In the context of this paper, problems with that theory as a normative theory are more important. In this section we first resolve these problems. Then we establish the normative validity of EU theory for decision-making by a trustee. Finally we consider possible alternative measures of risk.

2.3.1 Problems with EU Theory as a Normative Theory

One of the challenges to EU theory as a normative theory is the St Petersburg Paradox (Bernoulli, 1738), was discussed in Thomson (2003a), and precautions in the application of EU theory were suggested. Another is Rabin's (2000) theorem, which shows that:

> within the expected-utility model, anything but virtual risk neutrality over modest stakes implies manifestly unrealistic risk aversion over large stakes.

However, Cox & Sadiraj (2006) show that:

> the type of global small-stakes risk aversion assumed in … Rabin (2000) … has no implication for the expected utility of income model, hence no general implication for expected utility theory.

It appears that EU theory is not invalidated as a normative theory by either of these challenges. Nevertheless, if EU theory is to be used for normative purposes, care must be taken to ensure that the method of elicitation of utility functions does not fall foul of disparities between small and large ranges of outcomes. Also, where possible, outcomes should be measured in terms of income rather than in terms of wealth.

2.3.2 The Validity of EU theory as a Normative Theory

Raiffa (1961), Pratt (1964) and various subsequent authors contemplate the normative validity of EU theory, regardless of its descriptive validity. Pratt (op. cit.) refers to the "forging [of] a satisfactory utility [function] from more or less malleable preliminary preferences." Booth (unpublished) argues that:

the utility maximisation criterion provides a good model in which investment risk can be understood.

Thomson (2003a) argues that, while EU theory fails numerous tests of descriptive validity, its normative validity is not at issue. In particular, for a member of a retirement fund who subscribes to the axioms of expected utility theory, that theory has normative validity for the purposes of recommendations of investment-channel choice in a DC retirement fund. For a suitable axiomatisation of EU theory, reference may be made to that paper. By the same token, that theory has normative validity for the trustees of funds held on behalf of members and other prospective beneficiaries.

2.4 Other Measures of Risk

Other measures of risk may be considered for the purposes of asset allocation. For example, the capital-asset pricing model could be used. However, under EU theory it is not necessary to assume that the distribution of returns is elliptically symmetric.

Another example is the series of risk measures such as value at risk. Once again, though, EU theory is more general. For example, the 95% value at risk may be minimised by maximising a utility function with the argument:

$$z_T = \begin{cases} 0 \text{ for } A_T - L_T < k; \\ 1 \text{ otherwise;} \end{cases};$$

where:

A_T is the value of the assets at time horizon T;
L_T is the value of the assets at that time horizon;
and k is the lower 95% confidence limit of $A_T - L_T$.

For solvency risk, Haberman et al. (op. cit.) propose the use of the unconditional shortfall expectation. This may be minimised by maximising the expected value of a utility function with the argument:

$$z_T = \min\{A_T - L_T + D, 0\};$$

where D is a 'deterministic benchmark'; i.e. a threshold above which a shortfall is of concern.

For benefit value they propose the use of the discounted value of the expected future benefit payments and the expected liabilities at the time horizon. This may be maximised by maximising the expected value of the utility function:

$$z_T = L_T + P_T;$$

where:

L_T is the discounted value of the liabilities of the fund at time horizon T for subsequent reasonable expectations of benefits in respect of service to that time; and
P_T is the discounted value of payments made during the period $[0, T]$.

They also propose risk measures for contribution-rate risk and for the average employer contribution rate but, being of interest to the sponsoring employer rather than the trustees, these measures are outside of the scope of this paper.

2.5 The Application of EU Theory to Asset Allocation

In this section, the literature in which EU theory has been applied to asset allocation by pension funds is reviewed.

Sherris (1992) addresses the asset-allocation problem of a DB retirement fund. He shows how the mean–variance analysis of that problem by Wise (1984a, 1984b, 1987a, 1987b) and Wilkie (1985) may be generalised by means of EU theory. Cardinale et al. (2006) apply EU theory to the asset-allocation problem of a DB retirement fund.

Biasca (unpublished) also applies EU theory to the asset-allocation problem of a DB retirement fund or a hybrid between a DC and a DB retirement fund. Hoevenaars et al. (2008) apply EU theory to the asset-allocation problem of a financial institution with real long-term liabilities, citing a DB retirement fund as an example.

Numerous articles have been written about the use of EU theory for the asset-allocation problem facing a member of a DC retirement fund. As pointed out by Nielsen (unpublished), this is essentially the same as the asset-allocation problem facing an agent planning for retirement. Some of these (e.g. Booth, op. cit.; Boulier, Huang & Taillard, 2001; Cairns, Blake & Dowd, unpublished; Deelstra, Grasselli & Koehl, 2003; Battocchio & Menoncin, 2004; Khorasanee & Smith, unpublished; Nielsen, op. cit.) deal with the accumulation phase. Charupat & Milevsky (2002) deal with both the accumulation and decumulation phases. Others (e.g. Haberman & Vigna, 2001; Devolder, Bosche Princep & Dominguez Fabian, 2003; Gao, 2009) deal with the accumulation phase and a post-retirement annuity. Kingston & Thorp (2005) deal with the deferment of annuitisation after retirement.

The lack of application of EU theory to the asset-allocation problem in retirement funds appears to be due (at least in part) to the perception that it is not possible to elicit utility functions of members or of trustees (e.g. Jarvis, Lawrence & Miao, 2009). In principle there are valid objections to the elicitation of utility functions. These centre on the argument that, because EU theory is not a good descriptive theory, it cannot be assumed that the utility functions elicited from them will be a true reflection of their preferences. Nevertheless, elicitation methods can be chosen so as to reduce the distortions discovered by behavioural finance.

Another problem may be that the elicitation process is subjective—so subjective that decision-makers cannot be expected to be consistent in their responses to the elicitation process. However, it must be recognised that subjectivity is inevitable in defining attitudes to risk. It is arguably preferable to quantify that subjectivity in the elicitation of utility functions than to rely on the even greater subjectivity of type-1 prudence. Levitan & Thomson (2009) deal with the elicitation of utility functions of active members of a retirement fund.

Whilst the aforementioned articles on DC funds deal with long-term applications

to retirement funding, they all relate to the member's asset-allocation problem. They do not deal with the trustees' asset-allocation problem where there is no investment-channel choice, or where, amongst other investment choices the fund may offer, the trustees must determine a default channel or strategy. The utility functions required are therefore those of the individual members making such choices, not those of the trustees. If the trustees make systems available to elicit members' utility functions and recommend choices, however, they may consider it necessary to make trained personnel available to assist in the process (Thomson, 2003b).

Section 5 illustrates the application of EU theory to a DB fund. A similar approach may be used for the default channel of a DC fund.

3. DEFINITION OF THE UTILITY FUNCTION

In this section the definition of the utility function for the purposes of the asset-allocation decision is discussed. In sections 3.1 and 3.2 the objective variable used as the argument of the utility function is considered for DC funds and DB funds respectively. In section 3.3, the use of dynamic asset allocation is discussed. In section 3.4 the functional form of the utility function is considered. In section 3.5 the separation of value and risk is discussed, as well as the use of utility functions with discontinuities. In section 3.6, group decision-making is considered. The parameterisation of the utility function to allow for the required levels of type-2 prudence is discussed in section 3.7. Each of these subsections starts with (or in some cases comprises) a review of the literature on the subject concerned.

The discussion in this section is necessarily quite exhaustive. A reader more interested in the proposed class of utility functions may prefer to proceed to section 4.

3.1 The Argument of the Utility Function: DC Funds

In the economics literature (e.g. Von Neumann & Morgenstern, op. cit.; Savage, 1954; Samuelson, 1969) EU theory is developed as a basis for decision-making by individual agents. Its foundation lies in individual preferences relating to current and future consumption and investment, individual attitudes towards risk and individual bequest motives. It solves the asset-allocation problem in this context. The argument of the utility function is the consumption of the decision-maker, discounted at a rate equal to the subjective discount rate (Merton, 1969), which reflects the decision-maker's liquidity preference.

Samuelson (op. cit.) shows that, for constant relative risk aversion, the consumption–saving problem can be separated from the asset-allocation problem, so that, for the latter, utility can be expressed in terms of wealth instead of consumption. This separation presupposes constant prices of goods and services. Over time, wealth and investment returns must therefore be expressed in real terms.

Some actuarial authors—particularly in addressing the asset-allocation problem of a member of a DC retirement fund during the accumulation phase (e.g. Nielsen, op. cit.; also Merton, 1993; Deelstra, Graselli & Koehl, unpublished; Gerber & Shiu

(2000)—cf. Cairns, Blake & Dowd, op. cit.)—define utility functions in terms of terminal wealth, i.e. the lump-sum benefit on retirement. The problem with this definition is that it does not take cognisance of the post-retirement income that can be purchased from the benefit on retirement.

The use of terminal wealth is generally justified in terms of Samuelson (op. cit.). The problem with this is that, as observed above, the optimisation of asset allocation cannot be separated from the optimisation of consumption and savings, except for constant relative risk aversion. However, for an individual member of a DC fund during the accumulation phase, the consumption–savings problem does not arise, as the contribution rate is determined by the rules. (The optimisation of optional additional contributions is outside of the scope of this article.) Nevertheless, it should be recognised that the objective is to choose an asset allocation that will maximise post-retirement consumption—and therefore post-retirement income. Beyond retirement the consumption–savings problem does arise; it then becomes necessary to consider the retiring member's preferences between consumption and savings. This matter is further considered below.

Cairns, Blake & Dowd (op. cit.) use the pension purchased at retirement. The problem with this approach is that it treats members with large current balances as having greater expected utilities at retirement. Whilst this definition may be useful for a member, it does not solve the trustees' problem of designing a default asset allocation (i.e. of determining the asset allocation in the default investment channel) for all members of a certain attained age.

Vigna & Haberman (2001), Levitan (unpublished) and Levitan & Thomson (op. cit.) use the net replacement ratio as the argument of the utility function for the asset-allocation problem facing a member of a DC retirement fund. During the accumulation phase, members generally pay contributions in proportion to their earned income. Furthermore, it is reasonable to expect that members' post-retirement income requirements will be commensurate with their pre-retirement incomes. The use of a net replacement ratio may therefore be justified for the purposes of investment-channel choice by an individual member. It may also better describe a retiring member's preferences in terms of gains and losses of income at retirement, defined as in prospect theory (Kahneman & Tversky, op. cit.). However, it still does not address the trustees' problem of designing a default asset allocation for all members of a certain age. For a group of members of a particular attained age it would be possible to use an aggregate net replacement ratio for the whole group, but this would again place greater weight on those with large current balances than on those with small balances.

The above critique suggests that the argument of the utility function should be a ratio of the post-retirement income to a reference income. Since we are using EU theory to determine the optimal exposure to risky assets, we should use as our reference income a risk-free post-retirement income based on a member's current balance. Furthermore, in order to ensure that we can conveniently and fairly assign members to groups so that the same asset allocation will apply to each member of

a group, our argument should satisfy the requirement that, for each member of a particular group, the argument is the same.

For a member of a certain group, the argument of the utility function may be taken as the ratio of her/his post-retirement income to an inflation-protected deferred annuity notionally purchased from her/his current balance, expressed as:

$$z_T = \frac{A_T \cdot {}_{T-t}a_0}{A_0 . a_T}; \tag{1}$$

where:

A_T is the member's accumulated balance at time T, being her/his retirement date;

a_T is the price per unit, at time T, of an inflation-protected immediate annuity;

A_0 is the member's current balance (at time 0); and

${}_{T-t}a_0$ is the price per unit of an inflation-protected deferred annuity payable with effect from retirement, notionally purchased at time 0.

Since it is the same for all members of a particular attained age, with a particular retirement age and with particular annuity terms, the same ratio—and therefore the same asset allocation—may be used for all such members. Within each such group of members it also avoids placing greater emphasis on members with large current balances than on members with small balances. It therefore solves the trustees' problem of designing a default asset allocation: the same asset allocation may be used for every member in the group. This ratio is referred to below as the 'DC benefit ratio'.

The above definition presupposes that the benefit on retirement will be payable as a lump sum. Because preferences must ultimately be expressed in terms of goods and services rather than money, z_T is defined in terms of inflation-protected annuities. However, if and to the extent that the benefit on retirement is payable in the form of an annuity, the terms and conditions of that annuity (i.e. its frequency, its provision for a term certain or a survivor's annuity and whether it is inflation-protected) should be applied in the determination both of ${}_{T-t}a_0$ and of a_T.

Both A_T and a_T are stochastic. Allowance will have to be made for correlation between a_T and the returns on the various investment channels as these correlations will affect the variance of the distribution of z_T.

Depending on whether the fund buys annuities from a life office or pays them from its own assets, the value of a_T will have to be modelled accordingly. If a member has options between alternative annuities, or the option to defer annuitisation—or not to annuitise—then the trustees will need to decide on a default annuity option. This option may be applied before retirement, unless and until the member makes an option, for the purposes of simulating equation (1) during the accumulation phase. It may be appropriate to allow for a combination of alternative annuities, optimising between them by maximising the expected utility of real post-retirement income. For this purpose the bequest motive may be disregarded, because the trustees' fiduciary

duties are to the members and stated dependants, not to other legatees. As shown by Samuelson (op. cit.), though, the consumption–saving problem will not necessarily be separable, except for constant relative risk aversion. In general, the simultaneous maximisation of expected utility for annuitisation (and for asset allocation during deferment of annuitisation) and for post-retirement consumption year by year will therefore be complicated. In practice it would be preferable to assume no savings or dissavings—i.e. to assume that the accumulated savings will be immediately annuitised on attainment of the retirement age and that all post-retirement income will be required for consumption year by year. For this purpose a level inflation-protected annuity would need to be assumed for the default option. Even if most members select an annuity that is constant in nominal terms, the requirement of type-1 prudence suggests that the default should be expressed in real terms. There may be some room for compromise if allowance is made for life-cycle expenses varying by age, but this would also need to be expressed in real terms. The features of the default option—including frequency of payment and the provision of annuities to surviving spouses and dependants—would need to be reflected in the determination of a_T. There may be grounds for arguing that some capital may be required on retirement, either for paying off debts or for investment in income-generating activities. The counter-argument to this is that such arrangements are preferable only if and to the extent that they improve post-retirement income. The latter should therefore be used as the default argument.

In principle, A_t should include not only the retirement savings in the fund, but also those outside of the fund. If it is assumed that the retirement savings outside of the fund will be invested (and taxed) similarly to the asset allocation in the fund, the DC benefit ratio is unaffected. However, the trustees have no responsibility for members' retirement savings outside of the fund.

As shown in equation (1), the time horizon for a particular member is the date of attainment of the retirement age. However, it should be recognised that it will be possible to re-determine the default asset allocation at regular intervals during the accumulation phase. This necessitates dynamic asset allocation, as explained in section 3.3 below.

For the purposes of dynamic asset allocation, some authors (e.g. Cairns 1997) use continuous time. As explained by Cairns (op. cit.), this may be useful for the purposes of analysis of the sensitivity of asset allocations to various parameters. However, for practical applications the utility function and the models of risky assets may be too complex to permit solutions in closed form. In principle, as argued elsewhere by one of the authors (Thomson, 2011) the time interval to be used for dynamic asset allocation should be the same as the decision interval. Thus, if the trustees review the mandates of investment managers annually, if they reconsider their model of risky assets and the parameters of their utility function annually, and if members are grouped into annual age intervals for the purposes of determining default allocations, it would be appropriate to use an annual interval for dynamic modelling.

3.2 The Argument of the Utility Function: DB Funds

In considering the single-period asset-allocation problem of a DB retirement fund, Wise (1984a, b; 1987a, b) and Wilkie (op. cit.) use as their objective function the terminal surplus of the fund, i.e. the assets remaining after payment of all liabilities. Whilst they do not express their method in terms of expected utility, Sherris (unpublished) shows that it may be so expressed. He also shows that they effectively treat the problem as a single-period problem, the period being from the date of determination of asset allocation to the date of expiry of the liabilities. No reallocation of assets is contemplated during that period.

In considering the multiple-period problem, Sherris (unpublished) uses a recursive consumption-based utility function, treating surplus risk capital becoming payable to the sponsoring employer analogously to consumption. The problem with this approach is that it treats the sponsoring employer as the beneficiary of the pension fund. Sharpe & Tint (1990) use the objective variable:

$$z_T = A_T - kL_T \; ; \tag{2}$$

where:

A_T is the value of the assets at the time horizon T;
L_T is the value of the liabilities at that time horizon; and
k is a measure of the 'importance to be attached' to L_T.

Here they contemplate the possibility that $k = 0$; i.e. that the liabilities may be disregarded. That possibility is inconsistent with the requirements of prudence. For such purposes it would be necessary to let $k = 1$.

Boulier, Trussant & Florens (1995) and Siegman & Lucas (1999) use loss functions that effectively define the argument of the utility function as the discounted value of future contributions. This is clearly a reflection of the interests of the sponsoring employer rather than those of the trustees.

Cairns (op. cit.) defines a loss function that comprises one term relating to the discounted value of future contributions and one term relating to the discounted value of the future fund size from time to time. The former is intended to reflect the interests of the sponsoring employer and the latter the obligations of the trustees. Cairns, Blake & Dowd (op. cit.) use various loss functions involving both the discounted value of contributions and the value of the assets of the fund.

Haberman et al. (op. cit.) use various different 'performance measures', designed to be of interest to the various stakeholders they consider. Whilst they do not use utility functions, their performance measures may be expressed as the argument of a utility function. As a measure of solvency risk they propose the measure:

$$z_T = \min\left\{A_T - L_T + D, 0\right\} \; ; \tag{3}$$

where:

A_T is the value of the assets at the time horizon T;
L_T is the value of the liabilities at that time horizon; and
D is a 'threshold deficit' below which the deficit is not material.

Cardinale et al. (op. cit.) use a utility function that comprises a geometrically weighted average of two utility functions: one representing that of the sponsoring employer and the other that of the trustees as fiduciaries on behalf of members. The former is expressed in terms of contributions payable by the employer and the latter is identical to equation (3). Such a utility function may have value as a descriptive theory; in fact the authors discuss the relative weightings in terms of the "strength of the [respective] parties in bargaining." This means that such an approach cannot be regarded as a normative approach for the trustees as fiduciaries. In this paper it is assumed that the minimisation of the contribution rates required does not constitute an objective for the trustees. The trustees must, however, avoid the abuse of the trust that has been placed in them. This is discussed below.

Haberman et al. (op. cit.) argue that solvency risk is of particular concern to the trustees. This argument is based on the possibility of the wind-up of a scheme with insufficient assets. In the absence of insurance, that may be relevant. However, even then, a measure that explicitly relates to benefit value would be better; it is only to the extent that benefits to members are affected that solvency risk matters. As they point out, "the most important responsibility of the trustees is ensuring that the benefits promised by the scheme are paid." As noted in section 2.4, for benefit value Haberman et al. (op. cit.) propose the use of the total discounted value of the expected future benefit payments and the expected liabilities at the time horizon.

In a discussion on the question "Is this the end of the 'long term' for pensions actuaries?" held by the Faculty of Actuaries in 2005,[13] participants argued that, because DB liabilities in the United Kingdom were becoming guaranteed by sponsoring employers and, in the event of their failure, by the Pension Protection Fund, and because of other developments in the regulation of DB pension schemes, the role of the actuary was increasingly being driven by short-term objectives. In that discussion, the point was not made that, if liability-driven investment is adopted and if the trustees have constant relative risk aversion, then the asset-allocation decision will be the same for the short term as for the long term (Mossin, 1968). Under such conditions, a short-term focus is entirely justified. In this paper it is not necessarily assumed that trustees will follow constant relative risk aversion—indeed, as shown below, the requirement of type-2 prudence may suggest otherwise. However, one of the requirements of liability-driven investment is that long-term liabilities must be allowed for. Whether this is done by a life office for the purpose of buying out pensions or by the fund itself (or an investment manager on its behalf), the principle is unaffected: the problem of asset

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allocation requires long-term modelling of the liabilities. It is assumed in this paper that liability-driven investment will be used. Assumptions about risk aversion are dealt with below.

A more important aspect of that discussion related to the role of the trustees—in particular to the question whether pension-fund trustees in the UK still have a meaningful role to play. This matter is considered further below.

For a DB fund there is no reason why EU theory should not be expressed in terms of some variable other than terminal wealth (which in this context would generally be taken to mean the surplus at a specified time horizon). This would require that the axioms of the theory be expressed in terms of that variable. Provided the axioms are acceptable to the trustees when so expressed, the theory remains normatively valid.

Whilst for an individual the argument of the utility function may be defined in terms of her/his post-retirement income, it must be recognised that this is not necessarily appropriate for the trustees of a retirement fund. In the case of a DB fund the argument of the utility function must take cognisance not only of the assets of the fund but also of its liabilities. An individual has choices with regard to the liabilities that she/he may incur from time to time, but with the possible exceptions of increases to pensions and other improvements, the trustees have no such choices. The risks faced by the trustees of a retirement fund cannot be defined without reference to its liabilities.

For a DB fund, it would be natural to take 'wealth' to mean the excess of the assets over the liabilities; i.e. the surplus at a time horizon (as does Sherris (1992) in the single-period case). However, this would mean that the trustees would prefer a large fund to a small fund with the same funding ratio. That would clearly be wrong. This suggests that the funding ratio itself would be more appropriate. The funding ratio also has more meaning with regard to the security of the prospective beneficiaries.

A problem with the use of the funding ratio is that, if the argument of the utility function is required at the date of expiry of the liabilities, the denominator of the ratio is zero. Also—and more importantly—whilst in the absence of a sponsoring employer's guarantee and insurance against its failure a high funding ratio gives greater security to members, it does not relate to the benefits actually payable to members relative to their reasonable expectations.

The above critique suggests that the argument of the utility function should be expressed in terms of the benefits actually paid up to the time horizon, together with the assets available to pay subsequent benefits. As for the DC benefit ratio, the argument should be relative to a standard value. In contrast to the DC case, however, a DB fund creates reasonable expectations, which are unrelated to the contributions paid by and in respect of members. The standard value should therefore be expressed in terms of reasonable benefit expectations up to the time horizon, together with liabilities for subsequent reasonable expectations.

The argument may be taken as:

$$z_t = \frac{A_t + P_t}{\dot{L}_t + \dot{P}_t}; \tag{4}$$

where:

A_t is the value of the assets of the fund available at time t for subsequent benefits;
$\dot{L}_t$ is the value of the liabilities of the fund at time t for subsequent reasonable expectations of benefits in respect of service to time 0;
P_t is the value of payments actually made during the period [0, t], accumulated to time t with interest at the risk-free rate from time to time; and
$\dot{P}_t$ is the value of payments of reasonable benefit expectations during the period [0, t], accumulated to time t with interest at the risk-free rate from time to time.

In particular:

$$z_0 = \frac{A_0}{\dot{L}_0}; \tag{5}$$

and

$$z_T = \frac{A_T + P_T}{\dot{P}_T}; \tag{6}$$

where T is the date of expiry of the liabilities.

The numerator of equation (4) represents the value of benefits that have been paid up to time t and those that can be paid thereafter (or could have been paid up to time t in addition to those that have been paid). The denominator represents the reasonable expectation of benefits up to and beyond time t. We refer to the ratio as the 'DB benefit ratio'.

The above discussion relates only to past-service benefits, i.e. to benefits for service up to time 0. The optimal portfolio based on the maximisation of the expected utility of that argument relates to the assets of the fund at time 0. The argument for service currently accruing may be similarly determined. The optimal portfolio based on the maximisation of the expected utility of the latter argument relates to the investment of contributions to the fund during the period [0, 1) or the annual rate of contributions at time 0. This approach would correspond to the use of the projected-unit-credit method of valuation, which would be appropriate for an open fund. For a closed fund it would be appropriate to allow for all future service, which would correspond to an aggregate-funding method.

The use of the argument in equation (4) ignores the possibility that the sponsoring employer will restore the fund to a sound financial position or take a contribution holiday when the fund is in shortfall or deficit respectively. It also ignores the existence of any insurance against a shortfall in the event of the failure of the sponsoring employer. This is justified by the argument that, if the trustees were to take cognisance of such arrangements in their asset-allocation decision, it would constitute abuse of those arrangements. As shown by McCarthy & Miles (2011) the inclusion of

such arrangements in the asset-allocation decision results in considerably higher levels of risk-taking by the trustees. Although the trustees have a fiduciary duty towards the members, a normative approach to their asset-allocation decision would preclude such abuse. In a descriptive approach it would be necessary to treat the possibility of such abuse as a moral hazard; although the trustees are not agents either of the sponsoring employer or of the insurer, both of those parties to the arrangements are entitled to expect that, as a matter of good governance, the trustees will avoid such abuse and that any actuary advising the trustees will advise against it. The asset-allocation decision must therefore be made as if no such arrangements were in place.

The use of the argument in equation (4) also ignores the way in which, in the absence of interventions by the sponsoring employer and of insurance against its failure, benefits may be improved in the event of surpluses or reduced in the event of shortfalls from time to time. It is assumed that the effect of such improvements and reductions will be proportional to their effect on the numerator of equation (4) from time to time. More explicit assumptions could be made if so desired, but because the situation envisaged is likely to be quite hypothetical, it may be more appropriate to make a reasonable assumption such as that implicitly adopted here, namely that A_t represents the net benefit improvements that will have been made up to time t or may be made after that time. As and when payments are made as a result of such improvements, A_t decreases and P_t increases, so that the numerator is affected not by the improvement itself but by the emergence of excess assets that makes the improvement possible.

As for DC funds, it should be recognised that it will be possible to re-determine the asset allocation at regular intervals. This necessitates dynamic asset allocation. Again, the time interval to be used for dynamic asset allocation should be the same as the decision interval. Thus, if the trustees review the mandates of investment managers annually, if they reconsider their model of risky assets and the parameters of their utility function annually, and if the fund is valued annually, it would be appropriate to use an annual interval for dynamic modelling.

3.3 Dynamic Asset Allocation

As explained by Jarvis, Lawrence & Miao (op. cit.), the use of a long-term time horizon without allowance for dynamic asset allocation at shorter intervals is generally sub-optimal. This applies both to DC funds and to DB funds. An exception may occur if it is assumed that the trustees adopt constant relative risk aversion. However, as discussed below, that assumption is too restrictive and cannot be made in this paper.

As discussed in sections 3.1 and 3.2, the time interval to be used for dynamic programming will depend on the decision interval, both for DC funds and for DB funds. Without loss of generality it is assumed in this section that the decision interval is a year.

Haberman et al. (op. cit.) propose that dynamic asset allocation be allowed for by a specified investment strategy. Two alternative strategies are considered: an 'intuitive' strategy under which the proportion of the assets invested in equities

increases while the funding ratio increases and vice versa and a 'counter-intuitive' strategy under which the proportion of the assets invested in equities decreases while the funding ratio increases and vice versa. The problem with this approach is that the strategies are arbitrary. Whilst they illustrate a paradox, they are not necessarily optimal. The counter-intuitive strategy is particularly problematic in that it appears to be imprudent; this matter is discussed further below.

As shown by Mossin (op. cit.) we may allow for annual optimisation by dynamic programming as follows. Let $\boldsymbol{X}_{it}$ denote the ith simulation of a vector representing the state of the world at time t for $i = 1,\ldots,I$ and $t = 1,\ldots,T$. The components of that vector comprise the variables needed to determine the values of z_T from $\boldsymbol{X}_{iT}$ at time T (viz. A_T and a_T in equation (1) for a DC fund or A_T, P_T and $\dot{P}_T$ in equation (6) for a DB fund) as well as any other variables on which the value of $\boldsymbol{X}_{i,t+1}$ will depend (e.g. yield curves at time t and cumulative improvements in mortality to that time). We refer to these simulations as 'primary simulations'. For primary simulation i at time $t = T-1$ we may optimise for the final year prior to retirement by determining the maximum expected utility at that time, viz.:

$$u^*_{i,T-1}\left(\boldsymbol{X}_{i,T-1}\right) = \max_{\boldsymbol{g}}\left[E\left\{u\left(z_T \mid \boldsymbol{X}_{i,T-1}\right)\right\}\right];$$

where:

$\boldsymbol{g} = \begin{pmatrix} g_1 \\ \vdots \\ g_N \end{pmatrix}$ is a vector representing the proportion invested in each investment channel available, such that:

$$\sum_{n=1}^{N} g_n = 1 .$$

The function $u^*_{i,T-1}\left(\boldsymbol{X}\right)$ represents what Mossin (op. cit.) refers to as an 'indirect' or 'derived' utility function. For each primary simulation i the estimation of the value of that function requires a set of 'secondary estimations' $j = 1,\ldots,J$ of $\boldsymbol{X}_{i,T}$ (say $\boldsymbol{X}_{ijT}$), so that $\boldsymbol{g}$ may be calculated as the value that maximises:

$$\frac{1}{J}\sum_{j=1}^{J} u\left(z_{ijT} \mid \boldsymbol{X}_{ijT}\right).$$

That maximum then constitutes an estimate of $u^*_{i,T-1}\left(\boldsymbol{X}_{i,T-1}\right)$.

At time $t = T-2, T-3,\ldots,0$, as explained by Mossin (op. cit.) we may obtain the maximum indirect utility as:

$$u^*_{it}\left(\boldsymbol{X}_{it}\right) = \max_{\boldsymbol{g}}\left[E\left\{u^*_{it}\left(\boldsymbol{X}_{it}\right)\right\}\right].$$

As before, for that purpose we require secondary simulations. Here we assume that the function $u^*_{i,t}\left(\boldsymbol{X}\right)$ is known for all $\boldsymbol{X} \in \left(\boldsymbol{X}_{i1t},\ldots,\boldsymbol{X}_{iJt}\right)$. In fact we only have those values that have been simulated during the primary simulation, i.e. $\boldsymbol{X} \in \left(\boldsymbol{X}_{1t},\ldots,\boldsymbol{X}_{It}\right)$.

The required values must be determined either by interpolation or by fitting an indirect utility function.

Other authors apply dynamic asset allocation either in continuous time or by means of a recombinant tree or lattice (Jarvis, Lawrence & Miao, 2009). In discrete time, the use of recombinant trees or lattices is necessitated by the need to work forwards for primary simulations and backwards for secondary simulations; the same nodes are required for the latter as for the former. An alternative approach is to obtain estimates of indirect utilities for each primary node at the beginning of year t. For this purpose estimates of indirect utilities at each secondary node at the end of the year may be obtained by interpolation or smoothing between the estimates at each primary node at the end of the year. The estimate at the primary node at the beginning of the year is then the expected value of the utility at the end of the year across all the secondary nodes. This method is used in Thomson (2011). In this paper, however, we wish to use prudent utility functions for the maximisation of indirect expected utilities at the beginning of each year. This is because type-2 prudence must be applied not only to the asset allocation decision at the time horizon, but also to those at the beginning of each prior year. Instead of using interpolation or smoothing, we therefore fit a prudent utility function to the indirect utility function at the beginning of each year. This is done in the same manner as for the elicitation of such a utility function from trustees or members, as explained in section 4.2 below.

From the maximisation for $t=0$ we obtain an estimate of the value of $\boldsymbol{g}$ that represents the optimal asset allocation during the period [0, 1].

The values of I and J must be determined so that the process converges to a satisfactorily stable result.

3.4 The Functional Form of the Utility Function

Numerous empirical tests of the descriptive validity of various utility functions have been made. For example, Levy (2005) finds, in an experiment involving an actual monetary gain or loss indexed to the participant's investment performance, that "[decreasing absolute risk aversion] is strongly supported, but [increasing relative risk aversion] is rejected." Friend (1973) and Cohn et al. (1975) provide empirical evidence of decreasing relative risk aversion. Some evidence of decreasing relative risk aversion is also found by Projector & Weiss (unpublished) and Bossons (1973).

The shapes of the utility functions of individual decision-makers may be quite variable. Levitan & Thomson (op. cit.) show those elicited from a sample of members of a South African retirement fund. In that sample there are examples of individuals who show aspiration levels—i.e. points below which risk aversion is negative and above which it is positive. Such preferences would not be appropriate for fiduciaries as they constitute speculative behaviour. Indeed, any utility function that exhibits negative risk aversion over any part of its range would constitute speculative behaviour. Thus, the value function defined in prospect theory would typically be inappropriate over outcomes below current wealth. Furthermore, it may be argued that any utility

function that exhibits increasing absolute or relative risk aversion (depending on which of these measures is adopted) over any part of its range would constitute speculative behaviour; it would suggest that lower risk aversion may be acceptable at lower values of the benefit ratio than at higher values. Also, the elicitation of a utility function may be affected by the problems associated with the descriptive application of EU theory. These arguments are considered further below.

Mossin (op. cit.) shows functional forms of the utility function for which either 'complete' or 'partial myopia' applies. 'Complete myopia' means that the asset-allocation decision can be made as if the immediate period were the final one. He shows that, if the utility function is isoelastic, then complete myopia applies. 'Partial myopia' means that the asset-allocation decision can be made as if the immediate decision were the last one. Beyond the immediate period, the outcome would still have to be projected to the final time horizon in order to solve the asset-allocation problem, but it could be projected at the risk-free rate for all subsequent periods, thus simplifying the problem. He shows that, in order for partial myopia to apply, the utility function must belong to the HARA class. In practice, because of liability cash-flows, the condition of myopia does not help beyond the immediate period—say the year ahead. For that period it helps only if we make the simplifying assumption that all liability payments are made at the beginning or end of the year ahead. During that year, myopia applies, but beyond that year the liabilities change, so that the asset allocation will also have to change.

Mossin (op. cit.) is based on the presupposition that the argument of the utility function is wealth. The question arises whether either of these conditions applies when the argument of the utility function is a benefit ratio. Because the liabilities of a DB fund are specified by a process that is at least partially independent of the constituents of the market portfolio, the DB benefit ratio could not be predicted even if the return on each of the asset categories during the year were known. Furthermore, in a DB fund, the liabilities cannot themselves be treated as a short position in another asset category, because the allocation to that asset category is beyond the powers of the trustees. For these reasons, the asset-allocation problem is quite different from that contemplated by Mossin (op. cit.) and the conclusions reached there do not apply. Even in a DC fund the problem is different from Mossin's (op. cit.) because of the dependence of the DC benefit ratio on future mortality assumptions. In both cases we have an incomplete market.

For most purposes, the hyperbolic absolute risk aversion (HARA) class of utility functions has been considered to be sufficiently flexible and its mathematical tractability has established its popularity in the literature. As the name suggests, this class satisfies the requirement (Mossin, op. cit.) that, for a utility function $u(\bullet)$, the absolute risk aversion is:

$$r(x) = -\frac{u''(x)}{u'(x)} = \frac{1}{a+bx}\,. \tag{7}$$

For $a > 0$, $b > 0$, this establishes decreasing absolute risk aversion. Defining 'absolute risk tolerance' as

$$t(x) = \frac{1}{r(x)},$$

we obtain the result that absolute risk tolerance is linear. As shown by Merton (1971), the utility function for this class is:

$$u(x) = \begin{cases} \dfrac{\gamma}{1-\gamma}\left(\dfrac{\alpha x}{\gamma} + \beta\right)^{1-\gamma} & \text{for b} \neq 0, \gamma \neq 1; \\ \ln(x+\beta) \text{ for b} \neq 0, \gamma = 1; \\ -\exp(\alpha x) \text{ for } b = 0. \end{cases} \tag{8}$$

These forms are referred to as 'generalised power utility', 'generalised logarithmic utility' and 'exponential utility' respectively. This class includes:

— quadratic utility, with $\gamma = -1$;
— constant absolute risk aversion, with $b = 0$; and
— constant relative risk aversion with $b \neq 0, \beta = 0$.

HARA utility functions show decreasing relative risk aversion only if $\beta < 0$. This is generally problematic for small values of *x*. The problems with quadratic utility functions are well documented (e.g. Pratt, op. cit.; Booth, op. cit.) and are not further discussed here.

The utility function:

$$u(x) = \begin{cases} \dfrac{x^{1-\gamma} - 1}{1-\gamma} \text{ for } \gamma \neq 1 \\ \ln(x) \text{ for } \gamma = 1, \end{cases}$$

is referred to by Samuelson (op. cit.) as the 'isoelastic' utility function. The use of this expression is explained by Gerber & Shiu (op. cit.). Isoelastic utility implies (and is implied by) constant relative risk aversion:

$$\begin{aligned} \gamma(x) &= -x\frac{u''(x)}{u'(x)} \\ &= \gamma. \end{aligned} \tag{9}$$

Charupat & Milevsky (op. cit.) use constant relative risk aversion, apparently because of its tractability; they cite the solutions by Merton (1971; 1993) and Karatzas & Shreve

(1992) for the asset-allocation problem with constant relative risk aversion and a single risky asset whose price follows geometric Brownian motion. Nielsen (op. cit.) uses constant relative risk aversion because of its myopic quality, which not only yields the 'intuitively appealing strategy' that the proportion held in the risky asset is constant (cf. Mossin, op. cit.), but also the consequence that wealth never becomes negative. Cardinale et al. (op. cit.) use constant relative risk aversion because of its association with the Cobb–Douglas utility function. They also test the sensitivity of their results to that choice by comparing them with results using constant absolute risk aversion. Biasca (op. cit.) also uses constant relative risk aversion because there are no solutions to the asset-allocation problem (as he defines it) for any other type of utility function. Mehra & Prescott (1985) use constant relative risk aversion to explain the large average risk premium on historical returns on equity. Their reason is "to [ensure] that the equilibrium return process is stationary."

Boulier, Huang & Taillard (op. cit.), Deelstra, Grasselli & Koehl (2003), Huang, Milevsky & Wang (2008), Van Binsbergen & Brandt (unpublished) Hoevenaars et al. (op. cit.) and Døskeland & Nordahl (2008) use constant relative risk aversion without explanation. Cairns, Blake & Dowd (op. cit.) use constant relative risk aversion for illustrative purposes in the accumulation phase for a retirement-fund member. Malamud, Trubowitz & Wüthrich (2008) use constant relative risk aversion because "it has become standard in the actuarial literature." They observe (with numerous examples) that:

> even though it is now a common belief that [constant absolute risk aversion] does not properly describe investors' behavior, it is still very popular because of its nice multiplicative properties.

For the practical purposes envisaged in this paper, such an argument is not valid.

Khorasanee & Smith (op. cit.) use a logarithmic utility function because it satisfies certain requirements specified by the authors. They note, however, that this function "gives insufficient disutility to low levels of wealth" and they accordingly modify it, using:

$$u(z) = p_z \ln(z); \tag{10}$$

where:

$$p_z = \begin{cases} 1 \text{ for } z \geq 1; \\ p^* \text{ for z<1; and} \end{cases}$$

$$p^* \geq 1.$$

Booth (op. cit.) also proposes a logarithmic utility function with a discontinuity at "the wealth point below which the investor is believed to suffer particularly adverse consequences." Although he does not state the utility function explicitly, it is apparent from the article that the utility function intended is:

$$u(z) = \begin{cases} \ln(z) \text{ for } z < 1; \\ \ln(z) + h \text{ for } z \geq 1. \end{cases}$$

The use of discontinuities at critical values of the argument is discussed in section 3.5 below.

Li (2007) uses an adapted version of constant relative risk aversion to allow for time-varying risk aversion.

LeRoy (1973) uses constant absolute risk aversion, because for the purposes of his application (the extension of the capital-asset pricing model to the multiperiod case) he finds it more tractable. Cozzolino (unpublished) uses constant absolute risk aversion because of its tractability, because its shape meets certain criteria and because it "gives a certainty equivalent which agrees with the mean–variance objective when the profit is normally distributed but is sensitive to the shape of the distribution when it is not normally distributed." Sherris (unpublished) uses constant absolute risk aversion because he considers it 'sensible' in comparison with the quadratic. Battocchio & Menoncin (op. cit.) use constant absolute risk aversion because it allows the separation of the argument (real wealth) from the other state variables used. Yang & Zhang (2005) and Wang (2007) use constant absolute risk aversion without explanation. Pézier & Scheller (2011) use constant absolute risk aversion as a first approximation to utility functions with absolute risk aversion varying with wealth.

For illustrative purposes, Siegmann & Lucas (op. cit.), Devolder, Bosche Princep & Dominguez Fabian (op. cit.) and Gao (op. cit.) use both constant absolute risk aversion and constant relative risk aversion without explanation. Kingston & Thorp (op. cit.) use a HARA function with detailed explanation; however, that explanation relates to consumption levels, which are not relevant to decision-making by trustees.

Some authors (e.g. Boulier, Trussant & Florens, op. cit.; Cairns, op. cit.) use quadratic utility functions for illustrative purposes. These are generally presented as quadratic loss functions. Cairns (op. cit.) also uses constant and absolute risk aversion (also presented as loss functions) for illustrative purposes.

Satchell, Damant & Hwang (2000) propose the utility function:

$$u(x) = x - \lambda \exp\left\{-\theta\left(x - x^*\right)\right\}\left(x - x^*\right)^2.$$

This is a generalisation of quadratic utility designed to penalise expected returns by exponentially weighted variance.

Some authors (e.g. Siegmann, 2011) use a variety of forms of utility function for the purposes of illustration.

For the purposes of determining the form of the utility function, it would be possible to elicit the utility functions of at least a sample of members. The functional form so elicited may be parametric or non-parametric. Whilst such utility functions might be informative, they could not be used in any mechanical way to construct a

utility function for use by the trustees. In the first place the trustees are required to be prudent, and secondly, if such a method had been contemplated, it would not have been necessary to appoint them. The problem of group decision-making is considered in section 3.6 below. The way in which the utility functions of a sample of members may be used to inform the development of a utility function for use by the trustees is illustrated in sections 3.7 and 4.2.

In view of the above discussion we first consider the HARA class. From equation (9) it follows that the relative risk aversion for HARA utility is:

$$\gamma(z) = -z\frac{u''(z)}{u'(z)} = \frac{z}{az+b}\,.$$

The derivative of this is:

$$\frac{d}{dz}\big(\gamma(z)\big) = \frac{b}{\big(az+b\big)^2}\,.$$

For our purposes we require $a \geq 0$; otherwise risk aversion becomes negative for large z. We also require that $b \geq 0$ or that the range of z must be restricted to $z > \frac{-b}{a}$; otherwise there is a discontinuity at $z = \frac{-b}{a}$. The range of z cannot be so restricted. This means that, unless $b = 0$ (in which case relative risk aversion is constant), relative risk aversion is increasing. This applies both to generalised power utility and to generalised logarithmic utility. The latter has been advanced by Rubinstein (1976), but neither of these forms is possible with an argument $z \in (0,\infty)$.

The reasons advanced in the literature (LeRoy, op. cit.; Cozzolino, op. cit.; Sherris, unpublished; Yang & Zhang, op. cit.; Wang, op. cit.; Pézier & Scheller, op. cit.) for the use of constant absolute risk aversion are not convincing. It appears that, apart from that of Battocchio & Menoncin (op. cit.), which does not apply here, no convincing case has been made in the literature for the use of constant absolute risk aversion.

The findings of Projector & Weiss (op. cit.), Bossons (op. cit.), Friend (op. cit.), Cohn et al. (op. cit.) and Levy (op. cit.) that, even amongst individual decision-makers, non-increasing relative risk aversion is supported suggests that constant absolute risk aversion is not appropriate for trustees. Instead, non-increasing relative risk aversion should be used. It would be speculative for trustees to take more risk when funding ratios are low than when they are high. Since relative risk aversion is the measure to be used, this means that increasing relative risk aversion (and therefore constant absolute risk aversion) would be speculative.

Another advantage of the use of relative risk aversion over that of absolute risk aversion is that the former is dimensionless. The dimension of absolute risk tolerance is that of the argument of the utility function. This means that the dimension of absolute risk aversion is the inverse of that of the argument of the utility function.

The advantage of a dimensionless measure is that it applies regardless of the size of the fund. If, as in equation (4), the argument is itself dimensionless, it appears at first sight that the problem falls away. However, it must be borne in mind that, for given values of reasonable expectations at time 0, the proposed argument is effectively expressed in terms of currency. The use of relative risk aversion ensures that the problem is truly dimensionless even for given values of the denominator of equation (4).

In the light of the above arguments, apart from the case of constant relative risk aversion, the HARA class is inappropriate.

The fitting of utility functions as contemplated by Thomson (2003b) and Levitan & Thomson (op. cit.) allows for degrees of risk tolerance that, whilst they might be acceptable for individual members, would not be appropriate for the purposes of decision-making by trustees as contemplated in this paper.

The utility function proposed by Satchell, Damant & Hwang (op. cit.) is problematic in that the range of the argument that satisfies the requirement of risk aversion is constrained so that, as for the generalised power and generalised logarithmic utility functions, it cannot be used for $z \in (0, \infty)$.

Because of the requirements of type-2 prudence, the relative risk aversion will invariably be greater than 1; this is discussed further in section 3.7 below. The case of constant relative risk aversion may be acceptable to the trustees, and there is no intrinsic reason why it should not be. However, if the trustees require decreasing relative risk aversion, or if they wish to effect a compromise between different levels of relative risk aversion amongst themselves, then another class of utility functions is necessary. That is considered in section 4 below.

3.5 Separation of Value and Risk and the Use of Discontinuities

The discussion in sections 3.1, 3.2 and 3.4 presupposes that there can be no confusion between the specification of the argument of the utility function and its functional form. A moment's reflection will, however, challenge that presupposition. For example, consider the following cases. In case 1 the argument is the force of return over the forthcoming year and the utility function is linear. In case 2 the argument is wealth at the end of the period and the utility function is logarithmic. In case 1 we have:

$$u(x) = \delta \,;$$

where δ is the force of return over the forthcoming year. In case 2 we have:

$$\begin{aligned} u(x) &= \ln\left[A_0 \exp\left(\delta\right)\right] \\ &= \ln\left(A_0 \right) + \delta; \end{aligned}$$

where A_0 is the decision-maker's wealth at time 0. Since utility functions are unique up to shifting and scaling, these utility functions are the same.

Another matter that may cause confusion between the argument and the form of the utility function is the separation of attitudes to value from attitudes to risk. If,

in the absence of risk, the trustees of a DB fund would prefer an increase to a funding ratio of 1.2 when the current ratio is 1.1 than an increase to 1.3 when the current ratio is 1.2 then they may be said to have decreasing absolute 'strength of preference' (Dyer & Sarin, 1982): they have stronger preference for an extra 0.1 in the funding ratio when the funding ratio is low than when it is high. Unlike utility, strength of preference is not defined with reference to probabilities of outcomes. Dyer & Sarin (1982) develop a number of utility functions such that:

$$\tilde{u}(x) = u_p(v(x)) \ .$$

In this formulation, $v(\bullet)$ is a 'value' function measuring strength of preference and $u_p(\bullet)$ is a utility function relative to that value. They refer to the risk aversion associated with $u_p(\bullet)$ as the 'relative risk aversion'. Since, in this paper, this expression is used as defined in equation (9), it is not used here. To avoid confusion, the expression 'risk aversion relative to preference' is used here. The value function may take forms similar to those of utility functions, so that the effective utility function $\tilde{u}(\bullet)$ will take a form compounded from those of its components. By the same token, absolute or relative strength of preference may be defined in the same way with reference to the value function as absolute or relative risk aversion is defined with reference to a utility function; viz.:

$$-\frac{v''(z)}{v'(z)} \text{ and } -z\frac{v''(z)}{v'(z)}$$

respectively.

Dyer & Sarin (1979) show that, on the basis of certain reasonable axioms, the strength-of-preference value functions of a group may be determined as a positive linear function of those of the members of the group. Group decision-making is considered further in section 3.6.

The use of a discontinuous utility function such as

$$u(z) = \begin{cases} \ln(z) \text{ for } z < 1; \\ \ln(z) + h \text{ for } z \geq 1 \end{cases}$$

(Booth op. cit.) is clearly inappropriate. Such a discontinuity would imply, for example, that, if (say) $h = 1$, then the change from $z = 0{,}99$ to $z = 1$ has the same value to the trustees as the change from 0,36 to 0,99, even though the latter change is 63 times the former. This will clearly induce speculative behaviour: a desperate all-or-nothing aspiration to the critical funding ratio.

Jarvis, Lawrence & Miao (2009) suggest various ways of allowing for a 'target' in the specification of the utility function. They suggest (in effect):

$$u(z) = \begin{cases} -\infty \text{ for } z < 1; \\ \dfrac{(z-1)^{1-\gamma} - 1}{1-\gamma} \text{ for } z \geq 1; \end{cases}$$

$$u(z) = \begin{cases} -\infty \text{ for } z < 1; \\ \dfrac{z^{1-\gamma} - 1}{1-\gamma} \text{ for } z \geq 1; \end{cases}$$

$$u(z) = \begin{cases} \dfrac{z^{1-\gamma} - 1}{1-\gamma} \text{ for } z < 1; \\ 0 \text{ for } z \geq 1; \end{cases}$$

or some combination of those functions, where z is the funding level at a time horizon. These functions effectively bring option effects into the trustees' asset-allocation decision. The problem with the first two is that they introduce all-or-nothing objectives into that decision, which is speculative, and therefore contrary to type-2 prudence. Also, they suggest that a small shortfall is as bad as a large one. The problem with the last is that it ignores the additional security given to members' reasonable expectations by a higher level of funding.

The binary Dickensian notion "Annual income £20, annual expenditure £19–19–6, result happiness; annual income £20, annual expenditure £20–0–6, result misery"[14] may be true in principle, but in practice one can always forego the last sixpence of expenditure (or borrow sixpence to cover it) at the end of the year. The more you have left over at the end of the year (or the less you borrow), the less you need for the next. The security of the members is continuous through the target, so the notion of a target is misdirected. A target may be induced by regulation, but that is another matter; the effects of regulatory requirements should be programmed into the asset–liability model used for the projection of the assets of the fund and the payment of benefits to members. It is the ultimate benefits to members that matter, not the problems imposed on the trustees by regulatory requirements. If the trustees focus on the possible inconveniences of the latter to themselves instead of the optimisation of the former they are failing in their fiduciary duties towards members.

In most of the above functions the discontinuity is in the utility function itself. The question arises, though, whether a discontinuity either in the marginal utility function $u'(z)$ (as in Jarvis, Lawrence & Miao's (op. cit.) third alternative) or in the relative risk aversion $\gamma(z)$ is appropriate. Usually the reason for such a discontinuity arises from a discontinuity either in the marginal strength-of-preference value function v' or in the relative strength of preference, rather than in the marginal utility or relative risk aversion relative to the strength-of-preference value. These distinctions are discussed, with illustrative examples, in this section.

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The utility function introduced by Khorasanee & Smith (op. cit.) is instructive in that it illustrates how value and risk may be confounded. The formula in equation (10) could be interpreted in one of two ways: it may represent an adjustment to utility or to strength-of-preference value. The way in which it is presented suggests that the authors contemplated it as a pragmatic adjustment to utility. They point out that:

> As we fall below [the] critical value [$z = 1$], [$\ln(z)$] becomes negative, so multiplying [$\ln(z)$] by a positive constant greater than one will make the utility function more negative for [$z < 1$].

Strength-of-preference value may be defined by:

$$v(z) = z^{p_z},$$

so that, with a logarithmic utility function relative to strength-of-preference value, the 'effective utility function' may be expressed as:

$$\begin{aligned}\tilde{u}(z) &= u_p\left(v(z)\right)\\ &= \ln\left(v(z)\right)\\ &= \ln\left(z^{p_z}\right)\\ &= p_z \ln(z);\end{aligned}$$

where $u_p(\bullet)$ is the utility function relative to strength-of-preference value, or the utility function 'per se'.

As indicated above, relative strength of preference may be defined in the same way with reference to the strength-of-preference value function as relative risk aversion is defined with reference to a utility function. Let us define the relative strength of preference as:

$$\kappa_z = -z\frac{v''}{v'};$$

where v is the value of the strength-of-preference value function; i.e.:

$$v = v(z).$$

Then we may define that value as:

$$v = z^{1-\kappa_z};$$

where:

$$\kappa_z = \begin{cases} 1 \text{ for } z \le 1; \\ \kappa^* \in [0,1) \text{ for } z>1; \end{cases}$$

giving constant relative strength of preference equal to 1 up to the critical funding ratio and κ^* beyond it. This follows Khorasanee & Smith (op. cit.) as discussed above, with some change in notation. This means that, if the utility function relative to strength of preference is continuous, then the effective marginal utility function has a discontinuity at $z = 1$.

For a given relative-risk-aversion function $\gamma(z)$ it may be shown (scaling so that $u'(1) = 1$) that:

$$u'(z) = \exp\{-w(z)\};$$

where:

$$w(z) = \int_1^z \phi(x)dx;$$

and

$$\phi(x) = \frac{\gamma(x)}{x}.$$

This means that, if $u'(\bullet)$ (and therefore $w(\bullet)$) is discontinuous at $z = 1$ (or at any other value of z) then $\phi(\bullet)$—and therefore $\gamma(\bullet)$—is infinite at that value of z. If the limit is $+\infty$ there is an infinite increase in relative risk aversion as the funding ratio reaches its critical value from below, whilst if the limit is $-\infty$ there is an infinite increase as the funding ratio leaves its critical value. As discussed in section 3.2, whatever measure of risk aversion is adopted, it should be non-increasing. A discontinuity in effective marginal utility at the critical funding ratio (or at any other value of z) is therefore inappropriate.

It may be argued that the trustees' strength-of-preference values have nothing to do with risk aversion per se and that the use of discontinuous marginal strength-of-preference value functions therefore does not expose the trustees to charges of imprudence. But whether and to what extent the effective utility function is affected by strength of preference as opposed to risk aversion per se is immaterial. The trustees' type-2 prudence must be judged by the combined effect. After all, if the trustees' utility function is elicited, then it will be the effective utility function that is elicited, not the utility function relative to strength-of-preference values.

For the reasons discussed above, it is inappropriate to adopt a strength-of-preference value function—or an effective utility function—with a discontinuity either in the value or marginal value at the critical funding ratio, or indeed at any other funding ratio.

3.6 Group Decision-making

Decision-making by trustees on behalf of a large group of prospective beneficiaries with different utility functions cannot necessarily be dealt with on the basis of standard EU theory. The essential problem in doing so relates to that of interpersonal comparisons of utility. Arrow's (1951) impossibility theorem shows that, on certain

reasonable assumptions, but without direct interpersonal comparisons, there is generally no method of obtaining a unique ranking of alternative decisions based on the preferences of the individuals involved.

However, Harsanyi (1955, 1975) shows, on the basis of three postulates, that, in the setting of public policy, the social welfare function—in effect the utility function of a society based on the utility functions of its members—to be used by an agent making decisions on behalf of that society is:

$$u(x) = \sum_{m=1}^{M} c_m u_m(x); \tag{11}$$

where $u_m(x)$ is the utility function of member $m = 1, \ldots, M$ of that society. The first postulate is that the personal preferences of all the members satisfy the axioms of EU theory. The second is that the social preferences of the decision-maker satisfy those axioms. The third is that if every member is indifferent between a certain prospect and a risky prospect then the decision-maker will be socially indifferent between them. Harsanyi (1975) refers to these postulates as 'individual rationality', 'social rationality' and 'individualism'. To the extent that individual rationality is acceptable, social rationality should follow and the first two postulates are normatively uncontroversial (Thomson, 2003a). Sen (1973) argues, by implication (as inferred by Harsanyi (1975)), that the individualism postulate is normatively inappropriate because the decision-maker has an obligation to compensate disadvantaged members. Harsanyi (1975) argues that Sen's position is irrational because the decision-maker "cannot have any rational motive for wanting to impose his [*sic*] own [social] preferences on the members…"

Regarding the weights c_m in equation (11), Harsanyi (1955, 1975) postulates that, from the decision-maker's point of view, the identification of a particular member with a particular utility function is arbitrary and, by symmetry, must have a probability (or at least a prior probability) of $\frac{1}{M}$. On this basis he shows that $c_m = 1$ for all m. The major problem with this argument (as recognised by Harsanyi (1975)) is that utility functions are invariant under positive scaling. As observed by Sen (1970), this necessitates interpersonal comparisons of utility. To deal with this, Harsanyi (1975) argues that the decision-maker should scale each member's utility function so as to equal those that the decision-maker would derive from various outcomes if the decision-maker's circumstances were the same as the member's. He accepts that such assessments are subjective, difficult and '[not] always very reliable'. However, he asserts that decision-makers are obliged to make them "as carefully and as knowledgably" as they can.

Keeney (1976) shows, on the basis of certain reasonable postulates, that the group utility for decision H is:

$$u(H) = \sum_{m=1}^{M} c_m E\{u_m(X \mid H)\};$$

where $c_m \geq 0$ for $m = 1, \ldots M$, provided that there are at least two members for whom

$c_m > 0$. This is a different approach in that the group utility function cannot be specified independently of the distribution of the outcome X. Again, it involves interpersonal comparisons of preferences. As that author acknowledges, though, interpersonal utility comparisons are not easy, even when made by a 'benevolent dictator'; when requirements of participatory group decision-making are imposed, it may become impossible. In the case of a retirement fund, which may have thousands of prospective beneficiaries, it would almost certainly be impossible.

It would be possible to elicit the utility functions of each of the trustees. The questions used in the elicitation process would have to be framed to the context of their fiduciary obligations, since it is not the trustees' own attitude to risk that would be required, but their attitude as fiduciaries. It is unlikely that every trustee will have the same utility function. It may nevertheless be possible to obtain a collective utility function either by consensus or by compromise. For this purpose it may be helpful to start with a generalised functional form. If neither consensus nor compromise is possible, then an approach involving Harsanyi's (1975) individualism postulate needs to be adopted. In view of the need for type-2 prudence, equation (11) cannot be applied to all members. It can, however, be applied to all the trustees.

Harsanyi's (1975) defence against Sen's (1973) position is untenable because the trustees do have "a rational motive for wanting to impose [their] own [social] preferences on the members…": they are required to be prudent. If they fail to meet this requirement they are in dereliction of their fiduciary responsibilities.

The question whether the trustees have an obligation to compensate disadvantaged members may be relevant to decisions regarding increases to pensions, particularly to the supplementation of low pensions (Thomson, unpublished), but it is not relevant to the applications considered in this paper. However, the requirements of socially responsible investment are relevant here. In general, the application of these requirements to the investment-allocation decision will imply the application of constraints on the companies and states in which the assets of the fund may be invested, or the engagement of the trustees in shareholder activism. Subject to those constraints, investment mandates and benchmarks may be determined by means of EU theory.

3.7 Levels of Risk Aversion and Prudence

As noted in section 2.1, the legislative and regulatory requirements of the countries considered there explicitly or implicitly require that fiduciary duties including prudence, and common law in this regard are well established. We now turn to the relationship between risk appetite and type-2 prudence. In terms of EU theory, risk appetite is measured in terms of risk aversion. Since we have accepted EU theory as normatively valid, we shall consider the relationship between risk aversion and type-2 prudence.

Biasca (op. cit.) uses illustrative levels of constant relative risk aversion of 10 to 30. These levels are selected without justification. Cardinale et al. (op. cit.) use illustrative

levels of constant relative risk aversion of 0 to 7. Examples of widely differing levels of constant absolute risk aversion are given by Pézier and Schiller (op. cit.).

If the trustees are to make a reasonable decision about the level of relative risk aversion they should adopt, it would arguably be appropriate for them to be informed by the levels of relative risk aversion exhibited by members of the fund. In doing so, they need to recognise that not all members will be prudent. Nevertheless, such information could at least be used as a check on their own utility functions elicited as described in section 3.6, so as to establish how they as trustees compare with members. For this purpose the trustees may elicit the utility functions of a sample of members.

This approach may be illustrated with reference to the sample of members of a retirement fund that was used in Levitan & Thomson (op. cit.). More details of that sample, and of the research conducted on it, are contained in Levitan (op. cit.). In those works, for each respondent *m*, an average relative risk aversion was found as:

$$\overline{\gamma}_m = \frac{1}{x_\omega - x_\alpha} \int_{x_\alpha}^{x_\omega} \gamma_m(x)dx\,;$$

where x was the member's net replacement ratio at retirement and $\gamma_m(x)$ was the relative risk aversion of the member as a function of the net replacement ratio x. Whilst the net replacement ratio differs from the benefit ratios used in this paper, the elicitation could be framed in terms of benefit ratios. The range of integration $[x_\alpha, x_\omega]$ was the range of net replacement ratios over which the member's utility function was elicited. For the purpose of calculating $\gamma_m(x)$, the member's utility function was elicited using the equally likely certainty equivalent (ELCE) method (Anderson, Dillon & Hardaker, 1977; Farquhar, 1984; Thomson, 2003b) at discrete values of x over the range. The values of the utility function for intermediate values of x were found by interpolation, using an interpolation method suggested in Thomson (2003b).

The cumulative distribution of the average relative risk aversion of members in this sample is shown in Figure 3. As discussed in Levitan & Thomson (op. cit.), two outliers have been ignored. The median was 2.90 and the upper extreme was 7.06.

Because the trustees need to be more prudent than most members, the relative risk aversion to be adopted by the trustees should be at least the median. Clearly, though, one would not expect them to adopt the upper extreme. The relative risk aversion must therefore be between 2.90 and 7.06.

Sinn (2002) purports to prove that populations of decision-makers who do not adopt logarithmic utility functions (i.e. who do not have constant relative risk aversion of 1) will ultimately become extinct. Whilst a thorough criticism of that proof is beyond the scope of this paper, it appears to be based on weak premises. In particular, his argument that "natural selection will… pick a preference that leads to the maximisation of [an] expected growth factor" is based on an alleged but unproven symmetry with decision-making under certainty. Even if his argument is accepted, the time scale over which extinction may be expected to take place is clearly very large in comparison with the lifetimes of a retirement fund or its trustees: these constitute

an institutionalised population of decision-makers whose normative levels of relative risk aversion are—and whose descriptive levels of relative risk aversion should be—considerably greater than 1. Agents with high levels of relative risk aversion show no sign of extinction. For the time being it may be safely accepted that retirement funds and their trustees are here to stay, even if their decision-making function is reduced to that of default strategies for members of DC funds who rely on their prudence.

However, the requirement of type-2 prudence does not only relate to risk aversion; it also relates to the levels of risk aversion exhibited over the range of possible outcomes. Analogously to the Pratt–Arrow definitions of absolute and relative risk aversion, Kimball (1990) defines a 'coefficient of absolute prudence':

$$p(x) = -\frac{u'''(x)}{u''(x)};$$

and a 'coefficient of relative prudence':

$$\pi(x) = -x\frac{u'''(x)}{u''(x)}.$$

He motivates these coefficients by a definition of 'prudence' as "the sensitivity of the optimal choice of a decision variable" (e.g. the amount to be invested in the risk-free asset) to risk. He suggests that, if $p(x) > 0$ or $\pi(x) > 0$ then the decision-maker exhibits absolute or relative prudence respectively. He acknowledges, however, that, "in different contexts, 'prudence' will have different meanings." (ibid.) In the actuarial literature, this measure has been followed by Mazzoleni (unpublished) and Cardinale et al. (op. cit.).

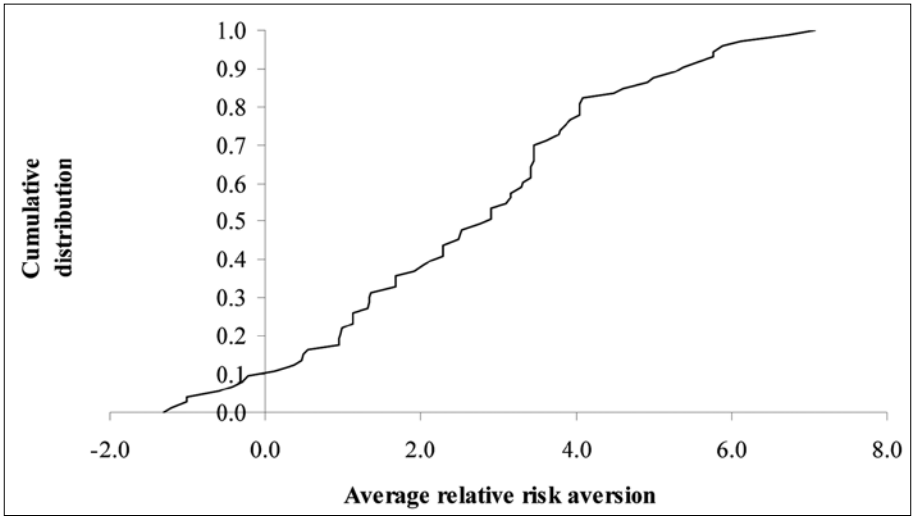

Figure 3 Cumulative sample distribution of average relative risk aversion

The requirements of due care and utmost good faith may be satisfied by ensuring that a coherent criterion for decision-making, such as that proposed in this paper, is adopted and disinterestedly applied to a reasonable model of the possible outcomes of the decision. The requirement of prudence is more difficult. The findings of Kimball (op. cit.), being couched in terms of 'prudence', are of relevance here, though, as that author points out, the definition of 'prudence' may differ in different contexts.

In the context of the obligations of trustees, it seems that Kimball's (op. cit.) definition is insufficient. If for example an isoelastic utility function is adopted by the trustees, then Kimball's coefficient of relative prudence will be:

$$\pi(z) = -z\frac{u'''(z)}{u''(z)} = \gamma(z) + 1 .$$

This will be positive (implying 'prudence') even for $-1 < \gamma < 0$. In that range a decision-maker would not even be risk-averse (and in fact not even unsatiated), let alone 'prudent' in the sense intended in the context of trustees' obligations.

In our context it is clearly intended that type-2 prudence relates to risk aversion. First it requires greater risk aversion than that of the average member, and certainly greater risk aversion than 1. Secondly, the requirement of type-2 prudence would be incompatible with increasing relative risk aversion, let alone increasing absolute risk aversion. It would not be prudent for a trustee to adopt lower risk aversion when the fund is in shortfall than when it is in surplus. (Some literature shows that, on certain assumptions, the adoption of this principle produces counter-intuitive results. That phenomenon is considered in section 4.3 below.) By the same token, as discussed in section 3.5, there should not be a discontinuity in marginal utility. Thirdly, for any given level of the argument of the utility function (the DC benefit ratio in the case of a DC fund or the DB benefit ratio in the case of a DB fund) the trustees' level of prudence should be greater than or equal to the level of prudence that would be exhibited by an agent with constant relative risk aversion equal to the trustees' relative risk aversion at that level of the argument.

In the light of the above discussion it follows that, in order to satisfy the requirements of type-2 prudence, the trustees' utility function $u(\bullet)$ should conform to the following criteria:

1) **The range criterion** It should map the open interval $(0, +\infty)$ into the open interval $(-\infty, +\infty)$ or into a subset of that interval.
2) **The continuity criterion** It should be at least twice differentiable and, if it is not thrice differentiable, then $u''(\bullet)$ should have a finite number of finite jump-discontinuities.
3) **The unsatiation criterion** The trustees should be unsatiated, so that, for all z, $u'(z) > 0$.
4) **The relative risk-aversion criterion** For all z there exists a γ^* such that $\gamma(z) \geq \gamma^* > 1$
5) **The non-increasing relative risk aversion criterion**
   (a) If over any range $u(\bullet)$ is thrice differentiable then, for all z in that range:

$A \quad \frac{d}{dz}\gamma(z) \le 0$; or

$B \quad \pi(z) \ge \gamma(z) + 1$.

(b) If $u'''(\bullet)$ is jump-discontinuous at any level of funding ratio z^* then:

A for any $\varepsilon > 0$ there exists $\Delta \in (0, \varepsilon)$ such that $\gamma\left(z^* - \Delta\right) \ge \gamma\left(z^*\right)$

or $\gamma\left(z^* + \Delta\right) \le \gamma\left(z^*\right)$; or

$B \quad \frac{\tilde{\pi}\left(z^*\right)}{\tilde{\gamma}\left(z^*\right) + 1} \ge 1$.

In condition 5(b)B, $\tilde{\pi}\left(z\right)$ and $\tilde{\gamma}(z)$ denote the limits of $\pi(z)$ and $\gamma(z)$ discussed below.

It may be shown as follows that, under condition 5(a):

$$A \Leftrightarrow B.$$

In the first place we have:

$$\begin{aligned}\frac{d}{dz}\gamma(z) &= \frac{d}{dz}\left(-z\frac{u''}{u'}\right)\\ &= -\frac{u''}{u} - z\frac{u'u''' - \left(u''\right)^2}{\left(u'\right)^2}\\ &= \frac{\gamma(z)}{z} - z\frac{u'''}{u'} + \frac{\left\{\gamma(z)\right\}^2}{z}.\end{aligned}$$

Since $z > 0$ and $\gamma(z) > 0$, we may multiply by $\frac{z}{\gamma(z)}$ to give:

$$\frac{d}{dz}\gamma(z) \le 0 \Leftrightarrow 1 - \frac{z^2 u'''}{\gamma(z)u'} + \gamma(z) \le 0. \tag{12}$$

Now:

$$\begin{aligned}\pi(z)\gamma(z) &= \left(-z\frac{u'''}{u''}\right)\left(-z\frac{u''}{u'}\right)\\ &= z^2\frac{u'''}{u'}.\end{aligned}$$

Substituting this into the right-hand inequality in (12) and rearranging, we have:

$$\frac{d}{dz}\gamma(z) \le 0 \Leftrightarrow \pi(z) \ge 1 + \gamma(z)\,;$$

i.e.:

$$A \Leftrightarrow B\,.$$

To prove that, under condition 5(b), $A \Leftrightarrow B$, we may construct a sequence of functions $u_n(z)$ that are thrice differentiable and converge monotonically to $u(z)$ as n tends to infinity. If we let:

$$\tilde{\gamma}(z) = \lim_{n\to\infty}\left(-z\frac{u_n''(z)}{u_n'(z)}\right); \text{ and}$$

$$\tilde{\tau}(z) = \lim_{n\to\infty}\left(-z\frac{u_n'''(z)}{u_n''(z)}\right);$$

then it may be proved that $A \Leftrightarrow B$.

It may be shown that conditions 5(a)A and 5(b)A both imply and are implied by the statement:

5. For any z^* and for all $z < z^*$, $\gamma(z) \ge \gamma(z^*)$.

For the sake of simplicity, condition 5 may therefore be replaced by this statement; i.e. $u(\bullet)$ exhibits non-increasing relative risk aversion.

In condition 4, following from the preceding discussion, the value of γ^* needs to be set high enough to establish the trustees' level of risk aversion as adequately prudent. On the other hand, the value of γ^* should not be unreasonably high.

4. DECREASING RELATIVE RISK AVERSION

As observed above, if the trustees require decreasing relative risk aversion, or if they wish to effect a compromise between different levels of relative risk aversion amongst themselves, then another class of utility functions is necessary. In section 4.1 such a class is defined and section 4.2 explains how to determine the parameters of utility functions of that class.

4.1 The WARRA Class

We define the WARRA class by a generalisation of the isoelastic utility function as follows:

$$u(z) = \frac{u_0(z) + cu_\infty(z)}{1+c}\,; \tag{13}$$

where:

$$u_0(z) = \frac{z^{1-\gamma_0} - 1}{1-\gamma_0};$$

$$u_\infty(z) = \frac{z^{1-\gamma_\infty} - 1}{1-\gamma_\infty};$$

$c > 0$; and

$\gamma_0 \geq \gamma_\infty > 1$.

The rationale behind the subscripts becomes clear below. The relative risk aversion of this utility function is:

$$\begin{aligned} \gamma(z) &= -z\frac{u''(z)}{u'(z)} \\ &= \frac{\gamma_0 z^{-\gamma_0} + c\gamma_\infty z^{-\gamma_\infty}}{z^{-\gamma_0} + cz^{-\gamma_\infty}}. \\ &= \frac{\gamma_0 + c\gamma_\infty z^{\lambda}}{1 + cz^{\lambda}}; \end{aligned} \tag{14}$$

where:

$\lambda = \gamma_0 - \gamma_\infty$.

The relative risk aversion of the WARRA class is thus a weighted average of those of a more risk-averse decision-maker and a less risk-averse decision-maker, the weighting increasing to the latter as the argument increases.

Figure 4 shows the relative risk aversion of a WARRA utility function as a function of the argument *z*. In that figure the following parameters have been used:

$\gamma_0 = 5$;

$\gamma_\infty = 3$; and

$c = 1$.

From equation (14) it may be shown that:

$$\lim_{z\to 0}\left(\gamma(z)\right) = \gamma_0; \text{ and}$$

$$\lim_{z\to \infty}\left(\gamma(z)\right) = \gamma_\infty.$$

This means that, if WARRA is adopted as a compromise between the most risk-averse trustee A, with constant relative risk aversion of γ_0, and the least risk-averse trustee B, with relative risk aversion of γ_∞, then, as shown in Figure 4, the relative risk aversion of trustee A is accommodated when funding levels are very low, and that of trustee B is accommodated when funding levels are very high. This provides a suitable basis for compromise, as trustee A is likely to be more concerned about the risk at low levels,

whilst trustee B is likely to be more concerned about excessive prudence at higher levels.

It may also be shown that:

$$\frac{d}{dz}\left(\gamma(z)\right) \leq 0\,;$$

thus establishing non-increasing risk aversion over the whole range, as shown in Figure 4. This shows that WARRA may be adopted for the purpose of deliberately achieving decreasing relative risk aversion.

For suitable values of the parameters, it follows that the WARRA class satisfies the requirements of type-2 prudence as set out in section 3.7. This class does not uniquely satisfy those requirements, but it provides a simple formulation of a utility function that does satisfy them, with sufficient flexibility for most purposes.

Finally, it may be shown that:

$$\gamma(1) = \gamma_0 + c\gamma_1\,;$$

so that, as shown in Figure 4,when the assets are equal to the liabilities, the weighting given to trustee B's risk aversion relative to trustee A's is equal to the weighting given to Trustee B's utility function relative to trustee A's.

4.2 The Determination of the Risk-aversion Parameters

If WARRA is adopted for the purpose of deliberately achieving decreasing relative risk aversion (as opposed to a compromise between less risk-averse and more risk-

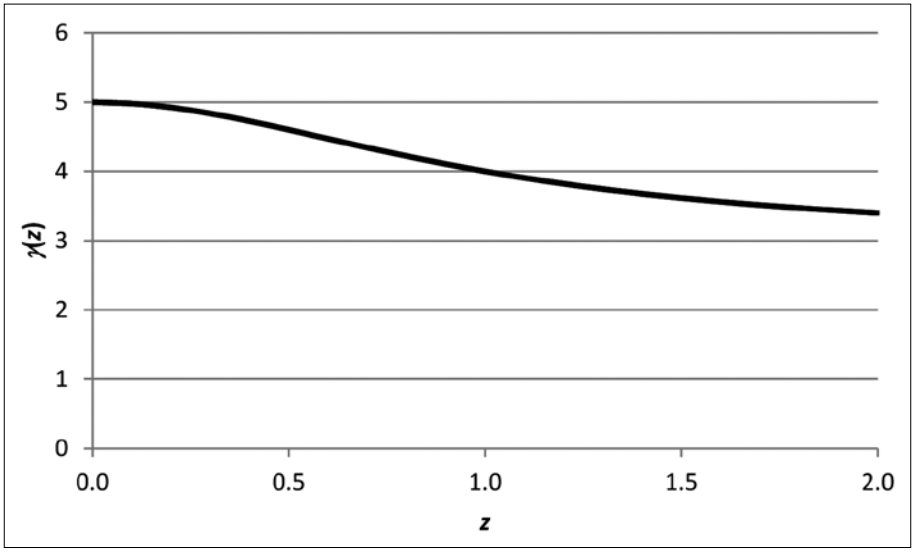

Figure 4 Relative risk aversion of a WARRA utility function

averse trustees) then the trustees must reach consensus as to relative risk aversion at extremely low funding levels (γ_0), at extremely high funding levels (γ_∞), and when the assets are equal to the liabilities, say:

$$\gamma(1) = \frac{\gamma_0 + c\gamma_\infty}{1+c};$$

so that:

$$c = \frac{\gamma_0 - \gamma(1)}{\gamma(1) - \gamma_\infty}.$$

For this purpose the trustees' utility functions at these levels need to be elicited. This may be done using the equally likely certainty equivalent (ELCE) method (Anderson, Dillon & Hardaker, op. cit.; Farquhar, op. cit.; Thomson, 2003b). This method addresses some of the problems raised above. If the trustees find it difficult to respond to the elicitation process with reference to extreme levels, a more reasonable range of levels may be used. Suppose the levels elicited are:

$$z_i \text{ for } i = 1,\ldots,5;$$

and that the corresponding values of the utility function are:

$$u_i \text{ for } i = 1,\ldots,5.$$

Without loss of generality we may assume that the utility function is shifted and scaled by the factors θ and ϕ, so that:

$$u_i = \theta + \phi v_i.$$

where v_i is the utility function before shifting and scaling, i.e.:

$$v_i = \frac{1}{1+c}\left(\frac{z_i^{1-\gamma_0} - 1}{1-\gamma_0} + c\frac{z_i^{1-\gamma_\infty} - 1}{1-\gamma_\infty}\right).$$

Let

$$w_i = \theta + \phi v_i - u_i.$$

Then we require

$$w_i = 0.$$

We may then express the relationships between the five observed values of z_i and u_i in terms of the simultaneous equations:

$$w_i = \theta + \frac{\phi}{1+c}\left(\frac{z_i^{1-\gamma_0} - 1}{1-\gamma_0} + c\frac{z_i^{1-\gamma_\infty} - 1}{1-\gamma_\infty}\right) - u_i \text{ for } i = 1,\ldots,5.$$

In order to solve these equations we may apply Newton's method as follows. Using the subscript n to denote the nth estimate of the respective parameters we have:

$$\boldsymbol{w}_n = \begin{pmatrix} w_{n1} \\ \vdots \\ w_{n5} \end{pmatrix} = \begin{pmatrix} \theta_n + \dfrac{\phi_n}{1+c_n}\left(\dfrac{z_i^{1-\gamma_{0n}}-1}{1-\gamma_{0n}} + c_n \dfrac{z_i^{1-\gamma_{\infty n}}-1}{1-\gamma_{\infty n}}\right) - u_i \\ \vdots \\ \theta_n + \dfrac{\phi_n}{1+c_n}\left(\dfrac{z_i^{1-\gamma_{0n}}-1}{1-\gamma_{0n}} + c_n \dfrac{z_i^{1-\gamma_{\infty n}}-1}{1-\gamma_{\infty n}}\right) - u_i \end{pmatrix}.$$

The Jacobian matrix of this equation is:

$$\boldsymbol{J}_n = \begin{pmatrix} \dfrac{\partial w_{n1}}{\partial \theta_n} & \dfrac{\partial w_{n1}}{\partial \phi_n} & \dfrac{\partial w_{n1}}{\partial \gamma_{0n}} & \dfrac{\partial w_{n1}}{\partial \gamma_{\infty n}} & \dfrac{\partial w_{n1}}{\partial c_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \dfrac{\partial w_{n5}}{\partial \theta_n} & \dfrac{\partial w_{n5}}{\partial \phi_n} & \dfrac{\partial w_{n5}}{\partial \gamma_{0n}} & \dfrac{\partial w_{n5}}{\partial \gamma_{\infty n}} & \dfrac{\partial w_{n5}}{\partial c_n} \end{pmatrix}.$$

where:

$$\frac{\partial w_{ni}}{\partial \theta_n} = 1;$$

$$\frac{\partial w_{ni}}{\partial \phi_n} = \frac{1}{1+c_n}\left(\frac{z_i^{1-\gamma_{0n}}-1}{1-\gamma_{0n}} + c_n \frac{z_i^{1-\gamma_{\infty n}}-1}{1-\gamma_{\infty n}}\right);$$

$$\frac{\partial w_{ni}}{\partial \gamma_{0n}} = \frac{\phi_n\left[z_i^{1-\gamma_{0n}}\left\{1-\left(1-\gamma_{0n}\right)\ln z_i\right\}-1\right]}{\left(1+c_n\right)\left(1-\gamma_{0n}\right)^2};$$

$$\frac{\partial w_{ni}}{\partial \gamma_{\infty n}} = \frac{\phi_n c_n\left[z_i^{1-\gamma_{\infty n}}\left\{1-\left(1-\gamma_{\infty n}\right)\ln z_i\right\}-1\right]}{\left(1+c_n\right)\left(1-\gamma_{\infty n}\right)^2}; \text{ and}$$

$$\frac{\partial w_{ni}}{\partial c_n} = \frac{\phi_n}{\left(1+c_n\right)^2}\left(\frac{z_i^{1-\gamma_{\infty n}}-1}{1-\gamma_{\infty n}} - \frac{z_i^{1-\gamma_{0n}}-1}{1-\gamma_{on}}\right).$$

Let:

$$\boldsymbol{\chi}_n = \begin{pmatrix} \theta_n \\ \phi_n \\ \gamma_{0n} \\ \gamma_{\infty n} \\ c_n \end{pmatrix}.$$

Applying Newton's method we may, in principle, solve the equation iteratively using the updating equation:

$$\boldsymbol{\chi}_{n+1} = \boldsymbol{\chi}_n - \boldsymbol{J}_n^{-1} w_n .$$

A problem arises with this approach, however: in general the Jacobian matrix is almost singular, so that Newton's method fails to converge. If we assume that the parameter c is known, then the above method may be applied, omitting that parameter. For that purpose we use only four observed values, we omit the last column and the bottom row of the Jacobian and the last component of $\boldsymbol{w}_n$ and $\boldsymbol{\chi}_n$. This procedure may be accessed by another function optimising c so as to approach a fifth observation as closely as possible. If c is allowed to become large, the higher risk aversion at lower levels of γ_0 would be undermined. In optimising c, it would therefore be appropriate to apply a maximum.

Algorithms for the determination of the parameters of the utility function are given in Appendix A. Some care needs to be taken with regard to the application of constraints; this is dealt with in that appendix.

Multiple solutions are possible. For this reason it may be necessary to use a range of starting values $\breve{\boldsymbol{\gamma}}_1$. It is also possible that no solution exists. If this is the case, it would be necessary to question the appropriateness of the trustees' attitudes to risk. If constant relative risk aversion is required the determination of the parameters of the utility function may be simplified accordingly.

The approach suggested in this section presupposes two postulates: first, that the trustees accept the axioms of EU theory as normative, and secondly that, the elicitation process does not introduce any bias into the measurement of trustees' risk aversion. As argued in Thomson (2003a), whilst the use of EU theory is not uniquely rational, it is rational. There is therefore no reason why it should not be accepted for normative purposes.

The second postulate, whilst problematic in principle, need not necessarily be regarded as problematic in practice. In the first place, the method of elicitation avoids—or at least reduces—the problems of framing, continuity, the common ratio condition, probability weighting, biases in subjective probabilities and the reference effect. Secondly, problems such as the Allais problem, preference reversal, inconsistent orderings, ambiguity and violations of the independence axiom may be reduced by education. Details of these problems have been described in Thomson (2003a). Other problems raised in the literature have also been addressed in that paper.

It should nevertheless be recognised that there may be some remaining bias, and the risk aversions determined should not be regarded as accurate; they should rather be interpreted as reasonably indicative.

5. COUNTER-INTUITIVE RESULTS AND THE CHALLENGE TO PRUDENCE

As argued in section 3.7, it would not be prudent for a trustee to adopt lower risk aversion when the fund is in shortfall than when it is in surplus. One would expect that, if the trustees' utility function shows higher relative risk aversion when the fund is in shortfall, then their exposure to the risky asset would be lower. Some literature contradicts this result. That literature is considered in section 5.1, and in section 5.2 the implications for the application of type-2 prudence are explored.

5.1 Literature Review of Counter-intuitive Results

Boulier, Trussant & Florens (op. cit.) assume that the liability of a pension fund is for total pensions equal to

$$p_t = e^{\alpha t} p_0$$

at time t. Two assets are available: a risk-free cash asset with a constant force of interest of r and a risky asset following Brownian motion. They use the quadratic loss function:

$$L(c_t) = c_t^2$$

where c_t is the contribution rate at time t. They determine the rate of contribution and the proportion invested in the risky asset that would minimise the value of

$$E\left\{\int_0^\infty e^{-\beta s} L(c_t)\,dt\right\};$$

where β is the decision-maker's subjective discount factor. For this purpose they use continuous-time optimisation. They find that, while the fund is in surplus, the assets should be invested in the risk-free asset, and while it is in shortfall, a proportion of the assets, increasing linearly with the amount of the shortfall, should be invested in the risky asset. This suggests that, for the case considered, a prudent approach would be strategically sub-optimal.

Siegmann & Lucas (op. cit.) extend Boulier, Trussant & Florens (op. cit.) to consider alternative loss functions, viz. a function exhibiting 'constant relative risk aversion' and one exhibiting 'constant absolute risk aversion'. Because of our adoption of relative risk aversion, the latter need not concern us here. The former is:

$$L(c_t) = \frac{c_t^{\gamma-1}}{\gamma-1};$$

where $\gamma > 0$. They find that the same results apply.

A problem with the above approaches is that the minimisation of contributions is not a concern to trustees. In fact high contributions from the sponsoring employer are advantageous to the security of members' benefits. The emergence of asset allocations that are counter-intuitive in terms of trustees' objectives is therefore not surprising.

Another problem is that they effectively attempt to apply optimal-consumption EU theory to negative consumption, which is a meaningless concept. The argument should be stated in terms of the sponsoring employer's overall objective, from which the pension-fund contributions would be a deduction.

Another problem with the above research is that it does not allow for randomness in the level of pension payments. Cairns (op. cit.) extends that research to include n risky assets as well as the risk-free asset, and he allows for random pension payments with a constant expected value. He uses a loss function $L(t, c, x)$ at time t expressed in terms of the contribution rate c and the value of the assets x at that time. He then determines the contribution rate and the asset allocation that would minimise the value of

$$E\left\{\int_t^{\infty} e^{-\beta s} L\big(s, c(s, X(s)), X(s)\big) ds \mid X(t) = x\right\};$$

where β is the decision-maker's subjective discount factor. He finds that the optimal proportion invested in each risky asset is constant relative to the others, so that the problem reduces to the case of one risky asset. Using a quadratic loss function, he finds that the optimal strategies c^* (for the contribution rate) and p^* (for the proportion of assets in the risky asset) depend only on x and not on t. In particular:

$$p^* = p_1^* + \frac{p_0^*}{x}.$$

In this expression, p_0^* is normally positive, so p^* is normally a decreasing function of x. Thus, as the funding level increases, the optimal exposure to the risky asset decreases, confirming Boulier, Trussant & Florens (op. cit.) and Siegman & Lucas (op. cit.). At a certain value of x, $p^* = 0$. That value represents a barrier through which X is unlikely to pass. With a constraint on the proportion invested in the risk-free asset, similar results are found, but effectively with two risky assets. He ascribes this result to the use of a quadratic loss function. He then considers power and exponential loss functions with constant pension payments. For the power loss function he effectively uses the utility function:

$$u(z_t) = \frac{z_t^{1-\gamma}}{1-\gamma};$$

where:

$$z = \begin{cases} c_m - c \text{ for } c \le c_m \\ -\infty \text{ for } c > c_m; \end{cases}$$

$$0 < \gamma < 1;$$

and c is the contribution rate.

Because of the discontinuity at the critical contribution rate c_m, this utility function does not fall within the criteria required for type-2 prudence. Nevertheless, he finds that, whilst (unlike the quadratic loss function) the solution is intuitive, the value of the assets tends to zero or infinity over time, depending on the decision-maker's subjective discount factor. The exponential loss function also does not conform to the requirements of type-2 prudence; like the quadratic loss function the solution is counter-intuitive.

He then considers 'continuous proportion portfolio insurance' (CPPI), under which the objective is to ensure that the value of the assets remains above a minimum level L_t. For this purpose he proposes an investment strategy in terms of which an amount equal to the value of the liabilities is invested in a low-risk portfolio (hedged against the liabilities) and an amount equal to the surplus is invested in a higher-risk portfolio. He shows that there is a static investment strategy that dominates the proposed CPPI strategy in the sense that the funding level has the same stationary mean but a lower stationary variance. However, if the trustees' objective were to minimise the probability that the funding level will fall below L_t then the proposed CPPI strategy would be better. The problem with this comparison is that, whilst the proposed strategy is heuristically appealing, it is not optimal in terms of dynamic investment optimisation under type-2 prudence.

Haberman et al. (op. cit.) suggest various objective variables. These include the discounted value of future benefits up to a time horizon and liabilities for subsequent benefits at that date. But in their actual presentation, on which their findings are based, they use only two: a 'mean shortfall risk' and an 'excess contribution risk'. The former is the excess (if any) of the assets over the liabilities at the time horizon and the latter is the discounted value of future adjustments to contributions to fund shortfalls up to the time horizon. They consider three strategies of asset allocation: a static strategy and two dynamic strategies. The assets comprise a long-dated fixed-interest gilt and equities. In the static strategy the asset allocation is fixed and annually rebalanced. That strategy need not be considered here. In the first dynamic asset strategy (the 'counter-intuitive' strategy) they assume that the asset allocations are changed at the end of every three years, as follows: for each 10% increase in the funding level (relative to the initial value), the allocation in equities is decreased by 5%, and vice versa. In the second (the 'intuitive' strategy) they assume that, for each 10% increase in the funding level (relative to the initial value), the allocation in equities is increased by 5%, and vice versa. The benefits are projected stochastically using a realistic model. They find that, in general, both the mean-shortfall risk and the excess-contribution-rate risk are lower for the counter-intuitive strategy than for the intuitive strategy.

Recognising that general conclusions cannot be drawn from illustrative examples, they consider a simplified case in which benefits are constant and the assets available are a risk-free cash asset and a risky asset. They show that, in this case, if the objective variable is the expected valuation deficit at a specified time horizon, the optimal asset allocation is counter-intuitive. For a quadratic loss function involving the value of the

assets and the contributions from time to time, they also obtain a counter-intuitive result. The problems with these results are that neither the arguments nor the form of the loss functions used conform to those required by trustees as discussed above and that the risk-free asset is not risk-free in relation to the liabilities.

In summary, the literature finds that, in a DB fund, for certain arguments and forms of the utility function, the asset allocation is counter-intuitive: at lower levels of funding, optimal investment allocations require greater exposure to risky assets when the fund is in shortfall than when it is in surplus. Because of the problems outlined above, it cannot be concluded that the dynamic use of a prudent utility function of the DB benefit ratio would necessarily produce counter-intuitive investment allocations. However, the circumstances (if any) in which type-2 prudence would produce such results must be considered.

5.2 The Challenge to Type-2 Prudence

Suppose for the sake of simplicity that there are two assets available: a risk-free asset matching the fund's liabilities for members' reasonable expectations and a risky asset. Let δ_t denote the force of growth of the liabilities (and therefore the force of return on the risk-free asset) during year t (where $t=1$ denotes the forthcoming year). Let p_t denote the amount of the benefits payable during year t, before allowance for growth to the beginning of the year. Then the amount of benefits payable during year t, before allowance for growth during that year, is:

$$\tilde{p}_t = \begin{cases} p_1 \text{ for } t=1 \\ p_t \exp\left(\sum_{s=1}^{t-1} \delta_s\right) \text{ for } t>1. \end{cases}$$

Suppose for convenience that the benefits paid during that year comprise the amount of $\frac{1}{2}\tilde{p}_t$ at the beginning of the year and the amount of $\frac{1}{2}\tilde{p}_t \exp(\delta_t)$ at the end. Suppose also that all payments of benefits follow reasonable expectations. p_t thus represents the value at the beginning of year 1 of the liability for the payment during year t.

From equation (4) the value of the DB benefit ratio at the end of year t will be:

$$z_t = \frac{\tilde{A}_t + \tilde{P}_t}{\tilde{L}_t + \tilde{P}_t} = 1 + \frac{\tilde{A}_t - \tilde{L}_t}{\tilde{L}_t + \tilde{P}_t}; \tag{15}$$

where:

$$\begin{aligned} \tilde{A}_t &= \left(\tilde{A}_{t-1} - \tfrac{1}{2}\tilde{p}_t\right)\exp\left(\delta_t + \alpha_t\lambda_t\right) - \tfrac{1}{2}\tilde{p}_t \exp\left(\delta_t\right) \\ &= \left\{\left(\tilde{A}_{t-1} - \tfrac{1}{2}\tilde{p}_t\right)\exp\left(\alpha_t\lambda_t\right) - \tfrac{1}{2}\tilde{p}_t\right\}\exp\left(\delta_t\right) \end{aligned}$$

is the value of the assets at the end of year t;

$$\tilde{P}_t = \left(\tilde{P}_{t-1} + \tilde{p}_t\right)\exp\left(\delta_t\right)$$

is the value at the end of year *t* of the benefits paid from year 1 up to and including year *t*;

$$\tilde{L}_t = \exp\left(\sum_{s=1}^{t} \delta_s\right) \sum_{s=t+1}^{T} p_s$$
$$= \left(\tilde{L}_{t-1} - \tilde{p}_t\right) \exp\left(\delta_t\right)$$

is the value at the end of year *t* of the liabilities in respect of benefits payable after that year;

α_t is the proportion of the fund's assets invested in the risky asset; and

λ_t is the force of return on the risky asset in year *t* in excess of that on the risk-free asset.

Since each variable both in the numerator $\tilde{A}_t - \tilde{L}_t$ in equation (15) and in the denominator $\tilde{L}_t + \tilde{P}_t$ is multiplied by $\exp\left(\delta_t\right)$, we may ignore that term and write:

$$z_t = 1 + \frac{A_t - L_t}{L_t + P_t}; \tag{16}$$

where:

$A_t = \left(A_{t-1} - \frac{1}{2} p_t\right) \exp\left(\alpha_t \lambda_t\right) - \frac{1}{2} p_t$;

$P_t = P_{t-1} + p_t$; and

$L_t = L_{t-1} - p_t$.

The values in equations (4) and (15) are as at time *t* whereas the values in equation (16) are discounted to time 0. Because the discounting in the numerator is the same as in the denominator, the ratio is the same. Also, in equation (4) the dots above the variables in the denominator indicate reasonable expectations. Here we assume that reasonable expectations will be met.

Furthermore, we may use the fact that:

$$P_T = L_t + P_t$$

to write:

$$z_t = \frac{A_t + P_t}{P_T}$$
$$= h_t + k_t \exp\left(\alpha_t \lambda_t\right); \tag{17}$$

where:

$h_t = \dfrac{P_t - \frac{1}{2} p_t}{P_T}$; and

$$k_t = \frac{A_{t-1} - \frac{1}{2} p_t}{P_T}.$$

In particular:

$$z_T = 1 + \frac{A_T}{P_T}.$$

The value of α_1 that gives the greatest indirect utility at the beginning of year 1 may be found by dynamic programming as explained in section 3.3. In this application, for a given value of z_{T-1} we may determine the value of α_T (say α_T^*) that maximises the expected utility at the end of year T. The expected utility for that value (say E_T^*) then constitutes the indirect utility for that value of z_{T-1}. We may determine such values for a range of values of A_{T-1} to give indirect utility function values $E_T^*(z)$. A WARRA-class utility function may then be fitted to these values to give parameters of the best fit. This procedure may be continued for t from T to 1. For $t = 1$ the known value A_0 of the assets may be used. It may happen that the resulting parameters will give lower relative risk aversion than the utility function used for year T. Whilst it may be acceptable to reduce the risk-aversion parameters of the indirect utility functions below those of the terminal values γ_{0T} and γ_∞, minimum values should be applied. The risk-aversion parameters are therefore subject to minima of $\gamma_{0\min}$ and $\gamma_{\infty\min}$ respectively. Also, the weighting parameter c is subject to a minimum of 0 and a selected maximum $c_{\max}$. An algorithm for the determination of α_1^* is given in Appendix B. R code is available from the authors.

For illustrative purposes the algorithm was used to determine the parameters of the indirect utility function and the optimum values α_1^* for the following parameters:

$\gamma_{0T} = 5$;

$\gamma_{\infty T} = 3$;

$c_T = 1$;

$\gamma_{0\min} = 4$;

$\gamma_{\infty\min} = 2$;

$\mu = 0.03$;

$\sigma = 0.1$;

$T = 20$;

$A_0 = 1$; and

$$\boldsymbol{p} = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_{20} \end{pmatrix} = \begin{pmatrix} \dfrac{40}{20\times 21} \\ \dfrac{38}{20\times 21} \\ \vdots \\ \dfrac{2}{20\times 21} \end{pmatrix}; \text{ so that } P_{20} = \sum_{t=1}^{20} p_t = 1 \text{ and } p_t \text{ decreases linearly.}$$

A_0 is thus expressed per unit of the value of the liabilities for future payments, the 'value' being the total of those payments before allowance for growth.

The values of γ_{0T}, $\gamma_{\infty T}$, $\gamma_{0\min}$ and $\gamma_{\infty\min}$ are informed by the values found in section 3.7.

Stopping conditions were used that generally produced reasonable convergence. It was found, however, that better convergence could be obtained by rerunning the program a few times and averaging the results.

The parameters of the indirect utility functions are shown in Figure 5. From that figure it may be noted that, as t decreases from T to 1 (i.e. following the curve backwards), the value of γ_{0t} ('gamma.0') initially decreases to a value close to the minimum of 4, after which it increases to about 7. The value of $\gamma_{\infty t}$ ('gamma.inf') shows a similar pattern. The value of c_t ('c') also reduces to a value close to its lower limit of 0, after which it increases to its limit of 1.5.

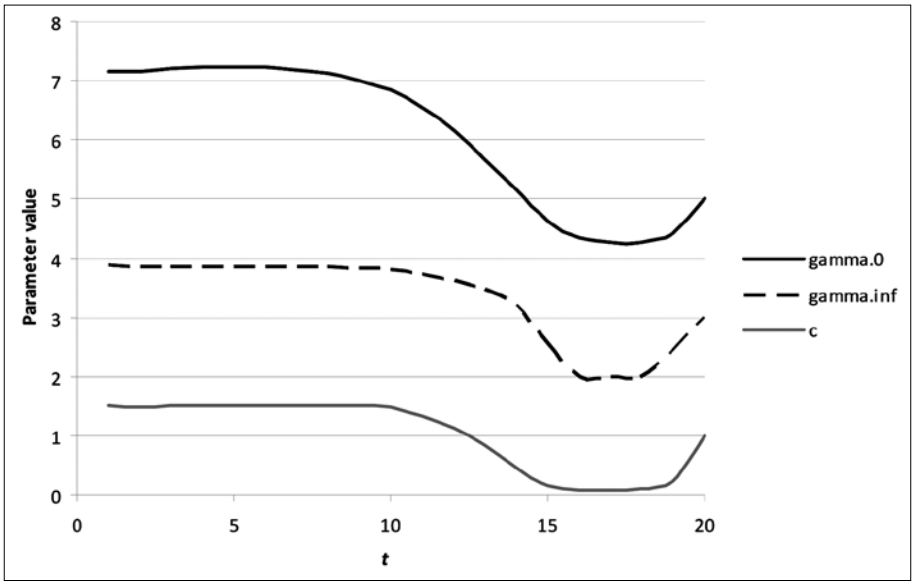

FIGURE 5 Parameters of the indirect utility functions

The resulting initial exposure to the risky asset was 0.754. This may seem high for risk aversion of up to 7. However, it should be borne in mind that:

— the assumption that $A_0 = P_T$ implies 100% funding on a risk-free basis, which is quite a strong financial condition;
— although the utility function itself belongs to the WARRA class and therefore reflects decreasing relative risk aversion, the argument of that function is the DB benefit ratio, which, as shown in equation (16), is a linear function of the exponent of the fund's return on the risky asset, and therefore incorporates features of the HARA class.

This means that, for the model of investment returns assumed, and for a strong fund with future liability payments decreasing linearly in real terms, even under the rigorous requirements of type-2 prudence specified by this paper, the optimal exposure to the risky asset is about 75%.

The sensitivity of this result to the assumptions was also tested. Figure 6 shows the sensitivity of the optimal exposure to the risky asset to the ultimate risk-aversion parameters γ_{0T} and $\gamma_{\infty T}$, ('gamma.0.T' and 'gamma.inf.T' respectively). For each parameter the result on the 'standard' set of assumptions is shown by a large square marker; the lines show results for alternative assumptions for that parameter, assuming that the other parameters are unchanged. Because of the limits of the various parameters relative to each other, the ranges of the various parameters are restricted; the intention is merely to give an indication of the effects of changes to the respective parameters one by one.

It may be seen from Figure 6 that increasing values tend to result in lower optimal exposures to the risky asset, which is intuitive.

The sensitivity of the optimal exposure to the minimum values $\gamma_{0\min}$ and $\gamma_{\infty\min}$ was found to be very low.

Figure 7 shows the sensitivity of the optimal exposure to the risky asset to the weighting parameters c_T and $c_{\max}$ ('c.T' and 'c.max' respectively).

Increases in c_T result in increases in the optimal allocation to the risky asset. This is also intuitive; higher values of c_T place greater emphasis on the lower risk aversion represented by $\gamma_{\infty T}$. This sensitivity is not great, because the values of c_t in earlier years are affected more by the optimal values of $\gamma_{\infty t}$ relative to those of γ_{0t} than by the ultimate value c_T. The optimal allocation is not very sensitive to $c_{\max}$; where the limit applies it tends to be offset by lower levels of $\gamma_{\infty t}$.

Figure 8 shows the sensitivity of the optimal exposure to the risky asset to the parameters μ ('mu') and σ ('sigma') of the distribution of the excess return on that asset.

These results are intuitive; as expected returns on risky assets increase, the exposure to them increases and as their volatility increases, exposure to them decreases. In fact, following the findings of Jones et al. (unpublished) regarding likely future investment returns under resource constraints, the lower levels of μ are more likely in the long term. If those findings are followed, the level selected will depend on

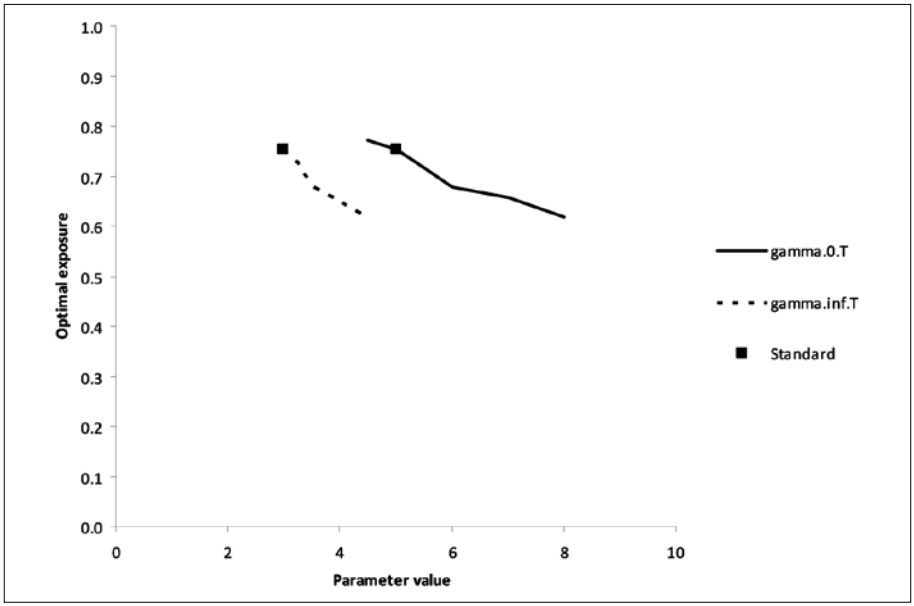

FIGURE 6 Sensitivity of the optimal exposure to the risky asset to the risk-aversion parameters of the utility function

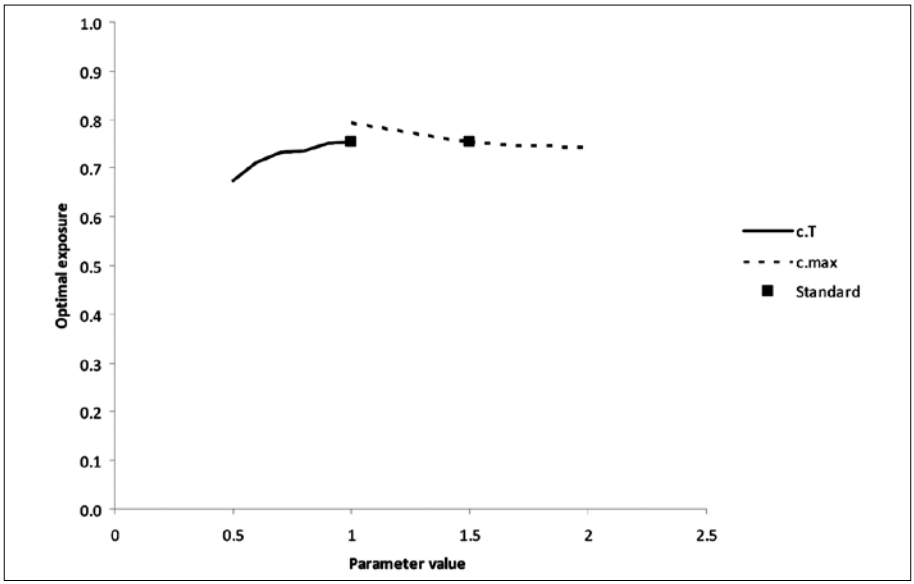

Figure 7. Sensitivity of the optimal exposure to the risky asset to the weighting parameters of the utility function

the scenario adopted by the trustees for decision-making. Furthermore, under certain scenarios, Jones et al. (op. cit.) contemplate considerably increased standard deviations of returns on investments. This will result in even lower optimal allocations to the risky asset.

Figure 9 shows the sensitivity of the optimal exposure to the risky asset to the term to expiry of the liabilities (T).

The downward trend is counter-intuitive; according to conventional wisdom one would expect that the expected advantages of investing in risky assets would emerge over the long term. The reason for this effect is that, for longer terms to redemption, the initial indirect risk-aversion factors are greater than for shorter terms.

Figure 10 shows the sensitivity of the optimal exposure to the risky asset A_0, i.e. the asset value at time 0, per unit of the value of the liabilities at that time. (Recall that the 'value' of the liabilities is the total of future payments before allowance for growth.) The 'standard' curve shows the results for the standard parameters used above. The 'high prudence' curve shows results with amended risk-aversion parameters and the 'high prudence, poor performance' curve shows results with further amendments to the mean and volatility of returns. The parameters used are shown in Table 1. The 'high prudence' and 'high prudence, poor performance' assumptions are explained below.

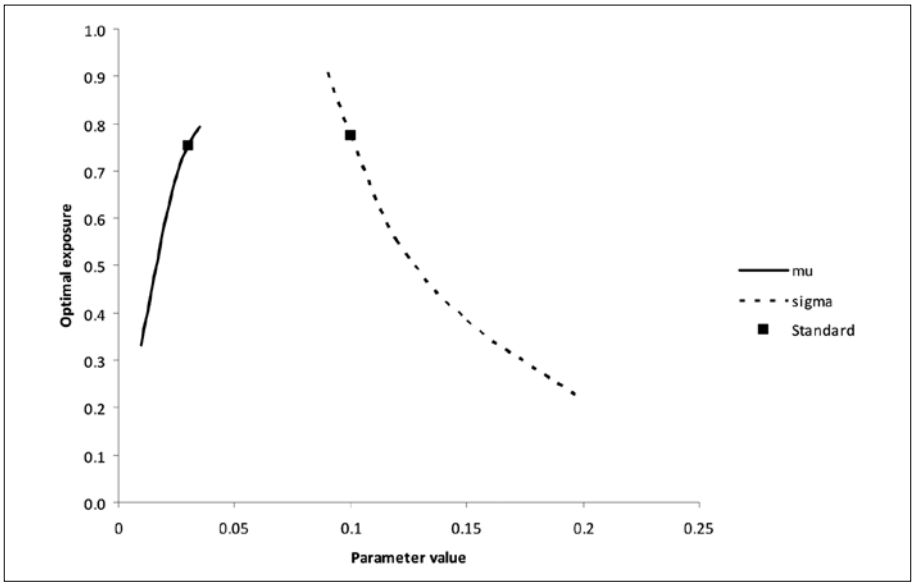

FIGURE 8 Sensitivity of the optimal exposure to the risky asset to the parameters of the distribution of the excess return on that asset

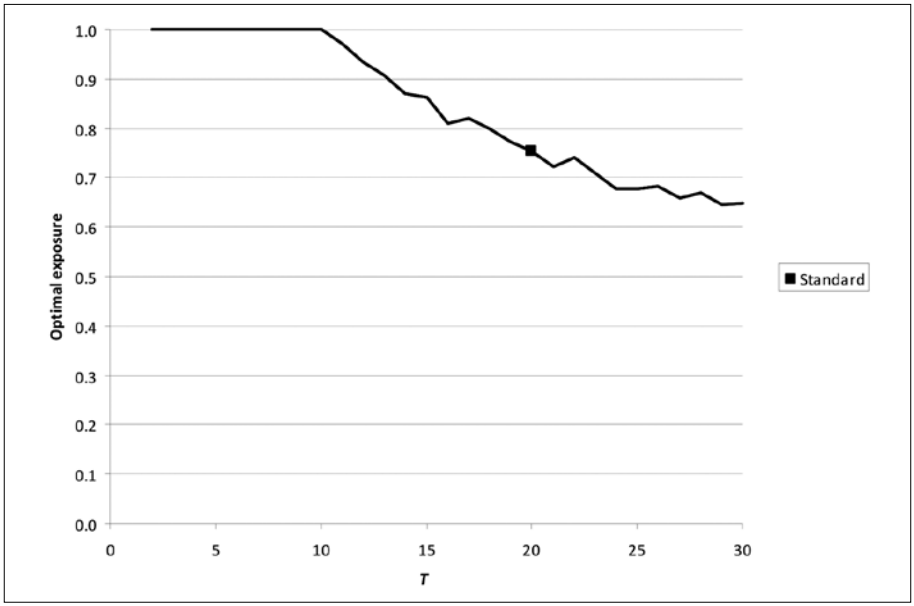

FIGURE 9 Sensitivity of the optimal exposure to the risky asset to the term to expiry of the liabilities

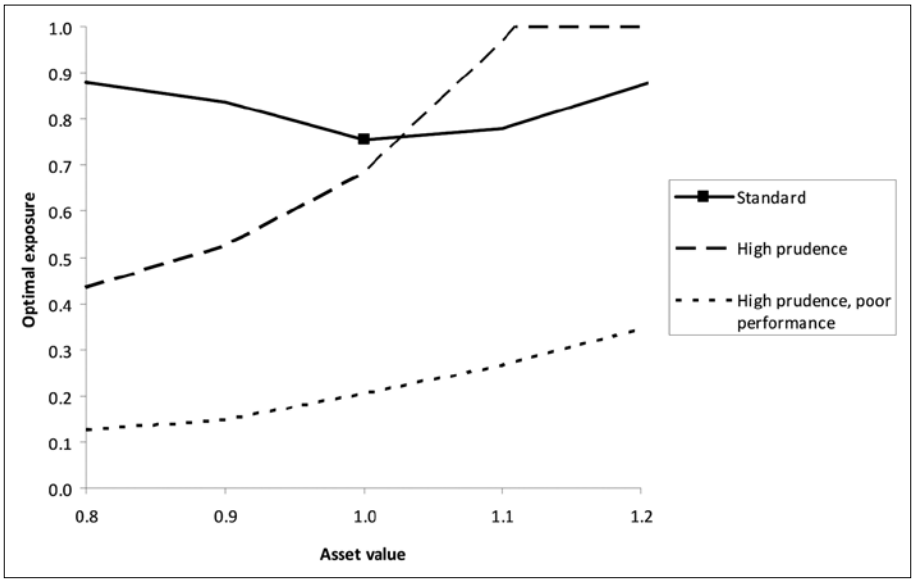

FIGURE 10 Sensitivity of the optimal exposure to the risky asset to asset value

Table 1 Parameters used in Figure 10

Parameter	Standard	High prudence	High prudence, poor performance
γ_{0T}	5	10	10
$\gamma_{\infty T}$	3	2	2
$\gamma_{0\min}$	4	6	6
$\gamma_{\infty\min}$	2	1.5	1.5
μ	0.03	0.03	0.02
σ	0.1	0.1	0.15

From the 'standard' curve it may be seen that, for levels of assets close to the value of total future liability payments, the optimal exposure is relatively low, whereas at lower and higher levels it is relatively high. For values of A_0 below 1, this result, like those of the articles referred to in section 3.8, is counter-intuitive: it suggests that if the fund is in deficit the trustees should invest more in risky assets than when it is fully funded. For higher values, this result differs from those of the articles referred to: it suggests that, following conventional wisdom, if the fund is in surplus the trustees have more freedom to invest in risky assets. These effects are not substantial; over the range of asset values considered, the range of exposure is only from 0.75 to 0.87. Nevertheless, it is counter-intuitive for relative asset values below 1.

Whilst it has often been observed (notably by Kahneman & Tversky, op. cit.) that investors invest more in risky assets below their aspiration levels, this appears to be speculative behaviour, which one would expect to be suboptimal under a utility function that is intended to reflect prudence.

The reason for the results at lower funding levels is that, at these levels, the initial indirect risk-aversion parameters are lower than for higher levels of funding. This may in itself be considered counter-intuitive; one might expect that, for an under-funded pension fund, the weighting of the indirect utility functions would be towards lower levels of ultimate outcomes, where the levels of relative risk aversion would be greater.

The technical explanation for this effect is that, in intermediate to later years, the term P_t in equation (17) (and therefore the term h_t) becomes more important than the term A_t (and therefore the term k_t). This means that the effect of exposure to risky assets is diluted. This effect establishes a heavier weighting to risky assets in those years than would otherwise be the case, and the indirect utility functions are similarly weighted. For a low initial level of funding, this effect is stronger, because the term P_t assumes greater importance at an earlier stage. This effect is carried through to earlier years to give the counter-intuitive result observed.

A more heuristic explanation for the effect is that the DB benefit ratio allows not only for the assets at a future date, but also for the benefits paid up to that date. For a fund currently in shortfall, the latter become more important than the former. This feature dilutes the effects of decreasing relative risk aversion.

However, when higher levels of prudence are considered the picture changes. For the 'high prudence' curve the risk-aversion parameters for low funding levels have been increased and those for high funding levels have been decreased. This means that relative risk aversion decreases at a higher rate as the DB benefit ratio at the time horizon increases. The result is intuitive: it suggests that the higher the funding level, the higher the optimal exposure to risky asset becomes. This effect becomes stronger than the effect of the dilution of the DB benefit ratio explained above.

This resolves the paradox in the findings of the articles discussed in section 3.8. Whether the level of prudence reflected in Table 1—or some level intermediate between the standard value and the high-prudence value—is justified is a matter of judgement, which should be informed with reference to members' levels of risk aversion.

If cognisance is taken of the effects of resource constraints, possible decreases in expected future returns and increases in volatility should be allowed for. The 'high prudence, poor performance' curve reflects such adjustment. At present there is considerable uncertainty about the future of investment returns under resource constraint. The use of models based on historical experience constitutes an assumption of business as usual, which is the worst scenario considered by Jones et al. (op. cit.). Clearly, the uncertainties about how models of future returns will differ from those of the past outweigh the uncertainties relating to risk-aversion parameters. The scenarios suggested by Jones et al. (op. cit.) comprise illustrative narratives. As such they do not constitute a scientific basis for the setting of assumptions. Until such time as the implications of resource constraints have been modelled, the adoption of the low-performance assumptions suggested in their analysis would constitute type-1 prudence. In the mean time the challenge to actuarial science is to develop models of future investment returns allowing for the effects of resource constraints.

6. SUMMARY AND SUGGESTIONS FOR FURTHER RESEARCH

6.1 Summary

In this paper it is argued that EU theory may be used as a normative theory for the purposes of asset allocation by the trustees of retirement funds.

For that purpose, the argument of the utility function for a DC fund should be the 'DC benefit ratio', defined as:

$$z_T = \frac{A_T \cdot {}_{T-t}a_0}{A_0 . a_T};$$

where:

A_T is the member's accumulated balance at time T, being her/his retirement date;
a_T is the price per unit, at time T, of an inflation-protected immediate annuity;
A_0 is the member's current balance (at time 0); and
${}_{T-t}a_0$ is the price per unit of the deferred annuity payable with effect from retirement, notionally purchased at time 0.

The argument of the utility function for a DB fund should be the 'DB benefit ratio', defined as:

$$z_t = \frac{A_t + P_t}{\dot{L}_t + \dot{P}_t};$$

where:

A_t is the value of the assets of the fund available at time t for subsequent benefits;
$\dot{L}_t$ is the value of the liabilities of the fund at time t for subsequent reasonable expectations of benefits in respect of service to time 0;
P_t is the value of payments actually made during the period [0, t], accumulated to time t with interest at the risk-free rate from time to time; and
$\dot{P}_t$ is the value of payments of reasonable benefit expectations during the period [0, t], accumulated to time t with interest at the risk-free rate from time to time.

It is argued that, in order to satisfy the requirements of type-2 prudence, the trustees' utility function $u(\bullet)$ should conform to the following criteria:

1) It should map the open interval $(0,+\infty)$ into the open interval $(-\infty,+\infty)$ or into a subset of that interval.
2) It should be at least twice differentiable and, if it is not thrice differentiable, then $u''(\bullet)$ should have a finite number of finite jump-discontinuities.
3) The trustees should be unsatiated, so that, for all z, $u'(z) > 0$.
4) For all z, $\gamma(z) \geq \gamma^* > 1$.
5) For any z^* and for all $z < z^*$, $\gamma(z) \geq \gamma(z^*)$.

A class of utility functions (the 'WARRA' class) that meets these requirements is introduced and its properties are discussed and shown to be satisfactory in terms of the above criteria. This class includes constant relative risk aversion as a special case.

The WARRA class is defined as:

$$u(z) = \frac{u_0(z) + cu_\infty(z)}{1+c};$$

where:

$$u_0(z) = \frac{z^{1-\gamma_0} - 1}{1-\gamma_0};$$

$$u_\infty(z) = \frac{z^{1-\gamma_\infty} - 1}{1-\gamma_\infty};$$

$c > 0$; and

$\gamma_0 \geq \gamma_\infty > 1$.

The relative risk aversion of this utility function is:

$$\gamma(z) = \frac{\gamma_0 + c\gamma_\infty z^\lambda}{1 + cz^\lambda}.$$

Methods of determining the risk-aversion parameters of a WARRA utility function are described with reference to values elicited from the trustees of a retirement fund. In particular, it is shown how, for a particular fund, they may be derived from the utility functions elicited from a sample of members of the fund.

In practice, spurious accuracy in the determination of the parameters of the trustees' utility function should be avoided. Sensitivity tests may be made so as to establish the extent to which marginal changes in those parameters will affect the decisions to be made. This may reduce the difficulties that trustees might experience in reaching consensus or a compromise with regard to the adoption of a utility function.

Counter-intuitive results found by other authors suggest that the trustees of a DB retirement fund should invest more in risky assets when the fund is in shortfall and less when it is in surplus. For moderate levels of prudence, it is found that, for levels of assets close to the value of total future liability payments, the optimal exposure is relatively low, whereas at lower and higher levels it is relatively high. For low funding levels, this result, like those of other articles, is counter-intuitive: it suggests that if the fund is in deficit the trustees should invest more in risky assets than when it is fully funded. Whilst it has often been observed that investors invest more in risky assets below their aspiration levels, this is generally perceived as speculative behaviour, which would be inappropriate for type-2 prudence. However, the explanation for this effect is that the DB benefit ratio allows not only for the assets at a future date, but also for the benefits paid up to that date. For a fund currently in shortfall, the latter become more important than the former. This feature dilutes the effects of decreasing relative risk aversion.

This means that the higher exposure to the risky asset at lower levels of funding is not speculative; it is merely a by-product of a well-justified focus on benefits payable to members rather than just on the assets (or the surplus) of the fund.

For high levels of prudence (i.e. for more strongly decreasing relative risk aversion) the paradox disappears: relative risk aversion decreases at a higher rate as the DB benefit ratio at the time horizon increases. The result is intuitive: it suggests that the higher the funding level, the higher the optimal exposure to risky asset becomes. This effect becomes stronger than the dilution effects. Whether such levels of prudence are justified is a matter of judgement, which should be informed with reference to members' levels of risk aversion.

If cognisance is taken of the effects of resource constraints, possible decreases in expected future returns and increases in volatility should be allowed for. At present there is considerable uncertainty about the future of investment returns under resource constraint. Clearly, the uncertainties about how models of future returns will differ from those of the past outweigh the uncertainties relating to risk-aversion parameters.

6.2 Suggestions for Further Research

Further research on levels of relative risk aversion both amongst members of retirement funds and amongst trustees would be helpful, particularly research on explanatory variables that might be used to inform or predict the parameters of WARRA utility functions for use by trustees. It would also be useful to establish the extent to which trustees are able to reach consensus or compromises with regard to the adoption of utility functions.

Further analysis of the implications of the proposed method for asset allocation by the trustees of retirement funds is also a matter for further research.

The application of WARRA utility functions to the pricing of the unhedgeable components of the liabilities of a DB retirement fund in an incomplete market could be explored. For this purpose, certainty equivalents could be used, which could be based on the trustees' utility function.

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APPENDIX A

ALGORITHM FOR THE DETERMINATION OF THE PARAMETERS OF A WARRA-CLASS UTILITY FUNCTION

Define the following functions:
(In these functions a breve (as in $\breve{\boldsymbol{\alpha}}$) refers to an argument of a function with respect to which that function is defined.)

(1) Determine the optimal parameter vector for a given value of *c* as follows:
For given values of:

$\breve{\boldsymbol{z}} = \begin{pmatrix} \breve{z}_1 \\ \vdots \\ \breve{z}_4 \end{pmatrix}$: the elicited values of the argument of the utility function;

$\breve{\boldsymbol{u}} = \begin{pmatrix} \breve{u}_1 \\ \vdots \\ \breve{u}_4 \end{pmatrix}$: the corresponding elicited values of the utility function;

$\breve{c}$: the parameter *c* of the utility function;

$\breve{\boldsymbol{\gamma}}_1 = \begin{pmatrix} \breve{\gamma}_{01} \\ \breve{\gamma}_{\infty 1} \end{pmatrix}$: the first trial values of the parameters γ_0 and γ_∞ of the utility function;

$\breve{\boldsymbol{\gamma}}_{\min} = \begin{pmatrix} \breve{\gamma}_{0,\min} \\ \breve{\gamma}_{\infty,\min} \end{pmatrix}$; the minimum values of the parameters;

$\breve{\eta}_\gamma$: the accuracy required for the stopping condition for the parameters γ_0 and γ_∞;

$\breve{N}_{\gamma\max}$: the maximum number of iterations; and

$\gamma_{\max}$: the maximum value of γ_0 and γ_∞ :

estimate the optimal parameter vector

$$\boldsymbol{\chi}\left(\breve{\boldsymbol{z}}, \breve{\boldsymbol{u}}, \breve{c}, \breve{\boldsymbol{\gamma}}_1, \breve{\boldsymbol{\gamma}}_{\min}, \breve{\eta}_\gamma, \breve{N}_{\gamma\max}\right) = \begin{pmatrix} \theta \\ \phi \\ \gamma_0 \\ \gamma_\infty \end{pmatrix}$$

of the WARRA-class utility function, given $\breve{c}$, as follows:

Let $n = 1$.

Let $\boldsymbol{\gamma}_1 = \breve{\boldsymbol{\gamma}}_1$.

While
$\{n = 1\}$; or
$\left\{\left(\left|\gamma_{0,n+1} - \gamma_{0,n}\right| > \breve{\eta}_\gamma \text{ or } \left|\gamma_{\infty,n+1} - \gamma_{\infty,n}\right| > \breve{\eta}_\gamma\right) \& \gamma_{0,n+1} < \gamma_{\max} \ \& \ I > 2 \ \& \ n \le \breve{N}_{\gamma\max}\right\}$:

Apply constraints and establish which parameters still need to be determined as follows. Here the binary variables $q_{\gamma 0}$ and $q_{\gamma\infty}$ indicate whether the variables γ_{0n} and $\gamma_{\infty n}$ need to be solved for.

Let $q_{\gamma 0} = 1$, $q_{\gamma\infty} = 1$.

If $\gamma_{\infty n} \le \gamma_{\infty\min}$ then $\gamma_{\infty n} = \gamma_{\infty\min}$ and $q_{\gamma\infty} = 0$.

If $\gamma_{0n} \le \gamma_{\infty n} + \breve{\eta}_\gamma$ then $\gamma_{0n} = \gamma_{\infty n}$ and $q_{\gamma\infty} = 0$.

If $\breve{c} = 0$ then $\gamma_{\infty n} = \gamma_{0n}$ and $q_{\gamma\infty} = 0$.

If $\gamma_{0n} \le \gamma_{0\min}$ then $\gamma_{0n} = \gamma_{0\min}$ and $q_{\gamma 0} = 0$.

Including θ_n and ϕ_n, for which we always have to solve, we define the number of unknown parameters as:

$$I = 2 + q_{\gamma 0} + q_{\gamma\infty}.$$

For $I = 2$ we use observations 1 and 4, for $I = 3$ we use observations 1, 2 and 4 and for $I = 4$ we use observations 1, 2, 3 and 4.

For each i in the observation set, determine the values of the utility function before shifting and scaling, as follows:

$$v_{0ni} = \frac{\breve{z}_i^{1-\gamma_{0n}} - 1}{1 - \gamma_{0n}};$$

$$v_{\infty ni} = \frac{\breve{z}_i^{1-\gamma_{\infty n}} - 1}{1 - \gamma_{\infty n}};$$

$$v_{ni} = \frac{1}{1 + \breve{c}}\left(v_{0in} + \breve{c} v_{\infty in}\right).$$

If $n = 1$ then determine the shifting and scaling factors as follows:

$$\phi_1 = \frac{\breve{u}_I - \breve{u}_1}{v_I - v_1};$$

$$\theta_1 = \breve{u}_2 - \phi_1 v_2;$$

For $n > 1$ θ_n and ϕ_n are obtained from the previous iteration.

For each i in the observation set, determine the error:

$$\Delta_{ni} = \theta_n + \phi_n v_{ni} - \breve{u}_i \,;$$

and hence the error vector:

$$\boldsymbol{\Delta}_n = \begin{pmatrix} \Delta_{n1} \\ \vdots \\ \Delta_{n4} \end{pmatrix}.$$

Determine the vector:

$$\boldsymbol{\chi}_n = \begin{pmatrix} \theta_n \\ \phi_n \\ \gamma_{0n} \\ \gamma_{\infty n} \end{pmatrix}$$

omitting γ_{0n} if $q_{\gamma 0} = 0$ and $\gamma_{\infty n}$ if $q_{\gamma\infty} = 0$.

For each i in the observation set, determine the following:

$$\frac{\partial \Delta_{ni}}{\partial \theta_n} = 1 \,;$$

$$\frac{\partial \Delta_{ni}}{\partial \phi_n} = \frac{1}{1+c_n}\left(\frac{z_i^{1-\gamma_{0n}} - 1}{1-\gamma_{0n}} + c_n \frac{z_i^{1-\gamma_{\infty n}} - 1}{1-\gamma_{\infty n}} \right);$$

$$\text{if } q_{\gamma 0} = 1 \text{ then } \frac{\partial \Delta_{ni}}{\partial \gamma_{0n}} = \frac{\phi_n \left[z_i^{1-\gamma_{0n}} \left\{ 1 - \left(1-\gamma_{0n}\right) \ln z_i \right\} - 1 \right]}{\left(1+c_n\right)\left(1-\gamma_{0n}\right)^2};$$

$$\text{if } q_{\gamma\infty} = 1 \text{ then } \frac{\partial \Delta_{ni}}{\partial \gamma_{\infty n}} = \frac{\phi_n c_n \left[z_i^{1-\gamma_{\infty n}} \left\{ 1 - \left(1-\gamma_{\infty n}\right) \ln z_i \right\} - 1 \right]}{\left(1+c_n\right)\left(1-\gamma_{\infty n}\right)^2};$$

and hence the Jacobian matrix:

$$\boldsymbol{J}_n = \begin{pmatrix} \dfrac{\partial \Delta_{n1}}{\partial \theta_n} & \dfrac{\partial \Delta_{n1}}{\partial \phi_n} & \dfrac{\partial \Delta_{n1}}{\partial \gamma_{0n}} & \dfrac{\partial \Delta_{n1}}{\partial \gamma_{\infty n}} \\ \vdots & \vdots & \vdots & \vdots \\ \dfrac{\partial \Delta_{n4}}{\partial \theta_n} & \dfrac{\partial \Delta_{n4}}{\partial \phi_n} & \dfrac{\partial \Delta_{n4}}{\partial \gamma_{0n}} & \dfrac{\partial \Delta_{n4}}{\partial \gamma_{\infty n}} \end{pmatrix},$$

omitting the columns for γ_{0n} if $q_{\gamma 0} = 0$ and for $\gamma_{\infty n}$ if $q_{\gamma\infty} = 0$, and omitting row 3 if $I \le 3$ and row 2 if $I = 2$.

Determine the next estimate of the parameter vector excluding known components:

$$\boldsymbol{\chi}_{n+1} = \boldsymbol{\chi}_n - \boldsymbol{J}_n^{-1}\Delta_n .$$

Let $n = n+1$.

Return the required parameter vector, including known components, as:

$$\boldsymbol{\chi} = \begin{pmatrix} \theta_{n+1} \\ \phi_{n+1} \\ \gamma_{0,n+1} \\ \gamma_{\infty,n+1} \end{pmatrix}.$$

(2) Determine the value x for which $y = f(x)$ is maximised, subject to the constraints $0 \le x \le x_{\max}$, as follows.
For given values

$$\breve{\boldsymbol{x}} = \begin{pmatrix} \breve{x}_1 \\ \vdots \\ \breve{x}_n \end{pmatrix} \text{ and } \breve{\boldsymbol{y}} = \begin{pmatrix} \breve{y}_1 \\ \vdots \\ \breve{y}_n \end{pmatrix};$$

where:

$$0 = \breve{x}_1 \le \breve{x}_2 \le \ldots \le \breve{x}_n = \breve{x}_{\max} ;$$

$$\breve{y}_i = f(\breve{x}_i) ;$$

and for a given value of $x_{\max}$, estimate the value

$$x = \psi\left(\breve{\boldsymbol{x}}, \breve{\boldsymbol{y}}, x_{\max}\right)$$

for which $y = f(x)$ is maximised, subject to the constraints:

$$0 \le x \le x_{\max} .$$

Select the value $j \in \{1,\ldots,n\}$ for which $\breve{y}_j$ is the greatest and select two values adjacent to j to give the selected vector:

$$\hat{\boldsymbol{x}} = \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{pmatrix} = \begin{cases} \begin{pmatrix} \breve{x}_1 \\ \breve{x}_2 \\ \breve{x}_3 \end{pmatrix} & \text{if } j = 1; \\ \begin{pmatrix} \breve{x}_{j-1} \\ \breve{x}_j \\ \breve{x}_{j+1} \end{pmatrix} & \text{if } 1 < j < n; \\ \begin{pmatrix} \breve{x}_{n-2} \\ \breve{x}_{n-1} \\ \breve{x}_n \end{pmatrix} & \text{if } j = n. \end{cases}$$

Select the corresponding values of $\breve{\boldsymbol{y}}$, viz.:

$$\hat{\boldsymbol{y}} = \begin{pmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \hat{y}_3 \end{pmatrix} = \begin{pmatrix} f(\hat{x}_1) \\ f(\hat{x}_2) \\ f(\hat{x}_3) \end{pmatrix}.$$

If $\hat{\boldsymbol{y}}$ is concave over $\hat{\boldsymbol{x}}$ then estimate the maximum value required by setting the derivative of the divided-difference formula equal to zero follows:

Let:

$$\begin{aligned} k_1 &= (\hat{x}_2 + \hat{x}_3)\hat{y}_1 \\ k_2 &= (\hat{x}_1 + \hat{x}_3)\hat{y}_2 \\ k_3 &= (\hat{x}_1 + \hat{x}_2)\hat{y}_3 \\ b_1 &= (\hat{x}_1 - \hat{x}_2)(\hat{x}_1 - \hat{x}_3) \\ b_2 &= (\hat{x}_2 - \hat{x}_1)(\hat{x}_2 - \hat{x}_3) \\ b_3 &= (\hat{x}_3 - \hat{x}_1)(\hat{x}_3 - \hat{x}_2) \end{aligned}.$$

Then:

$$x(\breve{\boldsymbol{x}}, \breve{\boldsymbol{y}}) = \frac{\dfrac{k_1}{b_1} + \dfrac{k_2}{b_2} + \dfrac{k_3}{b_3}}{2\left(\dfrac{\hat{y}_1}{b_1} + \dfrac{\hat{y}_2}{b_2} + \dfrac{\hat{y}_3}{b_3}\right)}$$

subject to a minimum of zero and a maximum of $x_{\max}$.

Otherwise let $x(\breve{\boldsymbol{x}}, \breve{\boldsymbol{y}}) = \breve{x}_j$.

(3) Determine the parameter vector of an elicited utility function, including parameter *c*, as follows:
For given values of:

$\breve{\boldsymbol{z}} = \begin{pmatrix} \breve{z}_1 \\ \vdots \\ \breve{z}_5 \end{pmatrix}$: the elicited values of the argument of the utility function;

$\breve{\boldsymbol{u}} = \begin{pmatrix} \breve{u}_1 \\ \vdots \\ \breve{u}_5 \end{pmatrix}$: the corresponding elicited values of the utility function;

$\breve{\boldsymbol{\gamma}}_1 = \begin{pmatrix} \breve{\gamma}_{01} \\ \breve{\gamma}_{\infty 1} \end{pmatrix}$: the first trial values of the parameters γ_0 and γ_∞ of the utility function;

$\breve{\boldsymbol{\gamma}}_{\min} = \begin{pmatrix} \breve{\gamma}_{0,\min} \\ \breve{\gamma}_{\infty,\min} \end{pmatrix}$; the minimum values of those parameters;

$\breve{c}_{\max}$: the maximum value of *c*;

$\breve{\eta}_\gamma$: the accuracy required for the stopping condition for the parameters γ_0 and γ_∞;

$\breve{N}_{\gamma\max}$: the maximum number of iterations in this function;

$\breve{\eta}_d$: the accuracy required for the stopping condition for the variable *d*;

$\breve{N}_{c\min}$: the minimum number of iterations in this function;

$\breve{N}_{c\max}$: the maximum number of iterations in this function;

estimate the optimal parameter vector

$$\boldsymbol{\chi}\left(\breve{\boldsymbol{z}}, \breve{\boldsymbol{u}}, \breve{\boldsymbol{\gamma}}_1, \breve{\boldsymbol{\gamma}}_{\min}, \breve{c}_{\max}, \breve{\eta}_\gamma, \breve{N}_{\gamma\max}, \breve{\eta}_d, \breve{N}_{c\min}, \breve{N}_{c\max}\right) = \begin{pmatrix} \theta \\ \phi \\ \gamma_0 \\ \gamma_\infty \\ c \end{pmatrix}$$

of the WARRA-class utility function as follows.

Determine the value of:

$$d_{\max} = \frac{\breve{c}_{\max}}{1 + \breve{c}_{\max}} .$$

$$\boldsymbol{d} = \begin{pmatrix} d_1 \\ \vdots \\ d_{\breve{N}_{c\min}} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ d_{\max} \end{pmatrix};$$

where:

$$d_m = \frac{(m-1)d_{\max}}{\breve{N}_{c\min} - 1}.$$

Determine the value of:

$$\boldsymbol{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_{\breve{N}_{c\min}} \end{pmatrix};$$

where:

$$c_m = \frac{d_m}{1-d_m}.$$

While $m \le \breve{N}_{c\min}$ and thereafter while $m \le \breve{N}_{c\max}$ and $d_m - d_{m-1} > \breve{\eta}_d$:

Using observations 1, 2, 4 and 5 determine the parameter vector

$$\boldsymbol{\chi}_m = \boldsymbol{\chi}\left(\breve{z}, \breve{\boldsymbol{u}}, \breve{c}, \breve{\boldsymbol{\gamma}}_1, \breve{\boldsymbol{\gamma}}_{\min}, \breve{\eta}_\gamma, \breve{N}\right) = \begin{pmatrix} \theta \\ \phi \\ \gamma_0 \\ \gamma_\infty \end{pmatrix}$$

by means of function (1).

Determine the following values:

$$v_{0m} = \frac{\breve{z}_3^{1-\gamma_{0m}} - 1}{1-\gamma_{0m}};$$

$$v_{\infty m} = \frac{\breve{z}_3^{1-\gamma_{\infty m}} - 1}{1-\gamma_{\infty m}};$$

$$v_m = (1-d_m)v_{0m} + d_m v_{\infty m}; \text{ and}$$

$$\Delta_m^2 = \left(\theta_m + \phi_m v_m - \breve{u}_3\right)^2.$$

If $m > \breve{N}_{c\min}$ then, using function (2), estimate the value of $d_m \in [0, d_{\max}]$ that maximises the value of $-\Delta_m^2$ and hence find $c_m = \dfrac{d_m}{1-d_m}$.

Let $m = m + 1$.

Excluding the last value of c_m, for which no value of Δ_m^2 has been calculated, find the value of c_m for which the value of Δ_m^2 is lowest.

Return the required parameter vector as:

$$\chi = \begin{pmatrix} \theta \\ \phi \\ \gamma_0 \\ \gamma_\infty \\ c_m \end{pmatrix}.$$

APPENDIX B

ALGORITHM FOR THE OPTIMISATION OF EXPOSURE TO A RISKY ASSET IN A DEFINED-BENEFITS PENSION FUND

Define the following parameters:

$\gamma_{0T}, \gamma_{\infty T}, c_T$: the parameters of the terminal utility function;

$\boldsymbol{\gamma}_{\min} = \begin{pmatrix} \gamma_{0\min} \\ \gamma_{\infty\min} \end{pmatrix}$: the minimum parameters of the indirect utility function;

$c_{\max}$: the maximum value of c;

μ, σ : the parameters of the distribution of the excess return on the risky asset;

T: the time horizon;

I: the number of primary simulations;

J: the number of secondary simulations;

ε: the accuracy required for the optimisation of exposure to the risky asset;

$N_{\alpha\min}$: the minimum number of iterations for that optimisation;

$N_{\alpha\max}$: the maximum number of iterations for that optimisation;

η_γ : the accuracy required for the optimisation of the parameters γ_0 and γ_∞;

η_d : the accuracy required for the optimisation of the parameter d;

$N_{c\min}$: the minimum number of iterations for that optimisation;

$N_{c\max}$: the maximum number of iterations for that optimisation;

A : the value of the assets at the start of year 1; and

P_T : the total future payments (to the end of year T) before allowance for growth.

Determine the following vectors:

$$\boldsymbol{p} = \begin{pmatrix} p_1 \\ \vdots \\ p_T \end{pmatrix};$$

where:

p_t is the amount of payment during year t before allowance for growth, i.e.:

$$p_t = \frac{2P_T(T-t+1)}{T(T+1)};$$

$$P_t = \sum_{s=1}^{t} p_s \ ;$$

$$\boldsymbol{h} = \begin{pmatrix} h_1 \\ \vdots \\ h_T \end{pmatrix};$$

where:

$$h_t = \frac{P_t - \frac{1}{2} p_t}{P_T} \text{; and}$$

$$\boldsymbol{\alpha}_{\text{init}} = \begin{pmatrix} \alpha_{\text{init}1} \\ \vdots \\ \alpha_{\text{init},N_{\alpha\min}} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}$$

where:

$$\alpha_{\text{init}\nu} = \frac{\nu - 1}{N_{\alpha\min} - 1} \ .$$

In the following steps products and sums of vectors and matrices are term by term.

Step 1:

This gives us a range of values of assets (before allowance for growth) assuming $\alpha_{it} = 1$.

Let $\boldsymbol{\Lambda} = \begin{pmatrix} \lambda_{11} & \cdots & \lambda_{1T} \\ \vdots & \ddots & \vdots \\ \lambda_{I1} & \cdots & \lambda_{IT} \end{pmatrix}$ be a pseudorandom sample from $N(\mu, \sigma)$ (the 'primary sample').

For $t = 1, \ldots, T-1$:

At the end of year t, we have:

$$\boldsymbol{A}_t = \left(\boldsymbol{A}_{t-1} - \tfrac{1}{2} p_t\right) \exp\left(\boldsymbol{\lambda}_t\right) - \tfrac{1}{2} p_t \,,$$

an I-component vector, being the value of the assets at the end of year t (before allowance for growth), where $\boldsymbol{\lambda}_t$ is the tth column of $\boldsymbol{\Lambda}$ and for $t = 1$, $\boldsymbol{A}_{t-1} = A$. $\boldsymbol{A}_t$ is sorted so that $A_{1t} \le A_{2t} \le \ldots \le A_{It}$.

$$\boldsymbol{k}_t = \frac{\boldsymbol{A}_{t-1} - \frac{1}{2} p_t}{P_T};$$

i.e.:

$$\begin{pmatrix} k_{1t} \\ \vdots \\ k_{It} \end{pmatrix} = \frac{1}{P_T} \begin{pmatrix} A_{1,t-1} - \frac{1}{2} p_t \\ \vdots \\ A_{I,t-1} - \frac{1}{2} p_t \end{pmatrix};$$

and:

$$\boldsymbol{z}_t = \begin{pmatrix} z_{1T} \\ \vdots \\ z_{IT} \end{pmatrix};$$

where:

$z_{it} = \dfrac{A_{it} + P_t}{P_T}$ is the ith simulation of the DB benefit ratio at the end of year t assuming $\alpha_{it} = 1$.

Let:

$$\boldsymbol{Z} = \begin{pmatrix} z_{11} & \cdots & z_{1,T-1} \\ \vdots & \ddots & \vdots \\ z_{I1} & \cdots & z_{5,T-1} \end{pmatrix}.$$

Select the five rows of $\boldsymbol{K}$ and $\boldsymbol{Z}$ for which $i = (.1I, .3I, .5I, .7I, .9I)$, rounded to the nearest integer. Renumber the rows to give:

$$\ddot{\boldsymbol{K}} = \begin{pmatrix} \ddot{k}_{11} & \cdots & \ddot{k}_{1T} \\ \vdots & \ddots & \vdots \\ \ddot{k}_{51} & \cdots & \ddot{k}_{5T} \end{pmatrix} \text{ and } \ddot{\boldsymbol{Z}} = \begin{pmatrix} \ddot{z}_{11} & \cdots & \ddot{z}_{1T} \\ \vdots & \ddots & \vdots \\ \ddot{z}_{51} & \cdots & \ddot{z}_{5T} \end{pmatrix};$$

where k_{it} and z_{it} are the ith selected values at the end of year t. This gives us five nodes at the end of each year for $t = 1, \ldots, T-1$.

Step 2:

This step gives us the optimal asset allocation and the corresponding expected value of the indirect utility function.

Let $\boldsymbol{\gamma}_0$, $\boldsymbol{\gamma}_\infty$ and $\boldsymbol{c}$ be T-component vectors representing the parameters of the indirect utility function at the end of year t for $t < T$ and those of the terminal utility function for $t = T$.

Let

$$\tilde{\boldsymbol{\Lambda}} = \begin{pmatrix} \tilde{\lambda}_{11} & \cdots & \tilde{\lambda}_{1T} \\ \vdots & \ddots & \vdots \\ \tilde{\lambda}_{J1} & \cdots & \tilde{\lambda}_{JT} \end{pmatrix}$$

be a pseudorandom sample from $N(\mu, \sigma)$ (the 'secondary sample').

For $t = T, \ldots, 1$:

If $t = T$ then we use the terminal utility function; i.e.:

$$\gamma_{0t} = \gamma_{0T};$$

$$\gamma_{\infty t} = \gamma_{\infty T};$$

$$c_t = c_T.$$

Otherwise from Appendix A function (3) we determine the parameters of the indirect utility function at the end of year *t*; i.e.:

$$\boldsymbol{\chi}\left(\boldsymbol{z}_t, \boldsymbol{E}^*_{t+1}, \gamma_{0T}, \gamma_{\inf T}, \boldsymbol{\gamma}_{\min}, \breve{c}_{\max}, \breve{\eta}_{\gamma}, \breve{\eta}_d, \breve{N}_{c\min}, \breve{N}_{c\max}\right) = \begin{pmatrix} \theta_t \\ \phi_t \\ \gamma_{0t} \\ \gamma_{\infty t} \\ c_t \end{pmatrix}.$$

(Note that $\boldsymbol{z}_t$ and $\boldsymbol{E}^*_{t+1}$ are $\ddot{I}$ -component vectors: $\boldsymbol{z}_t$, the DB benefit ratio at the end of year *t*, comes from step 1 and $\boldsymbol{E}^*_{t+1}$, the maximum expected utility at the beginning of year $t+1$, comes from below for the previous value of *t*. The values γ_{0T} and $\gamma_{\infty T}$, are used here as the initial trial values for the determination of γ_{0t} and $\gamma_{\infty t}$ at the end of year *t*.)

If $t = 1$ then $\ddot{I} = 1$, otherwise $\ddot{I} = 5$.

For $i = 1, \ldots, \ddot{I}$:

Let $\boldsymbol{\alpha} = \boldsymbol{\alpha}_{\text{init}}$.

Let $\nu = 1$.

While $\nu \le N_{\alpha\min}$ and thereafter while $|\alpha_\nu - \alpha_{\nu-1}| > \varepsilon$ and $\nu \le N_{\alpha\max}$:

If $\nu > N_{\alpha\min}$ then from Appendix A function (2) determine an updated estimate:

$$\alpha_\nu = \psi(\boldsymbol{\alpha}, \boldsymbol{E}, 1).$$

Determine:

$$\tilde{z}_\nu = h_t + k_{it} \exp\left(\alpha_\nu \tilde{\lambda}_t\right);$$

i.e. the DB benefit ratio at the end of year *t*, where $\tilde{\boldsymbol{\lambda}}_t$ is the *t*th column of $\tilde{\boldsymbol{\Lambda}}$;

$$\tilde{\boldsymbol{u}}_v = \frac{1}{1+c_t}\left(\frac{\tilde{z}_v^{1-\gamma_{0t}}-1}{1-\gamma_{0t}} + c_t\frac{\tilde{z}_v^{1-\gamma_{\infty t}}-1}{1-\gamma_{\infty t}}\right); \text{ and}$$

$$E_{itv} = \frac{1}{J}\sum_{j=1}^{J}\tilde{u}_{v,j}\ ;$$

i.e. the expected utility at the beginning of the year.

Let $v = v+1$.

Determine the optimal exposure and the maximum expected utility at the beginning of the year as follows:

Let $\alpha_{it} = \alpha_v$

Let $E_{it}^* = E_{it,v-1}$

The optimal allocation to the risky asset is α_{11}.