THE QUANTIFICATION OF TYPE-2 PRUDENCE IN ASSET ALLOCATION BY THE TRUSTEES OF A RETIREMENT FUND

Presented by: Rob Thomson

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The quantification of type-2 prudence

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2. General review of literature

3. Definition of the utility function

4. Decreasing relative risk aversion

5. Counter-intuitive results and the challenge to prudence

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Introduction

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   - they have fiduciary responsibilities towards beneficiaries;
   and
   - they are required to be prudent.
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   \[
   E = \int_{0}^{\infty} u(z) f(z) \, dz
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General review of the literature

- The requirement of prudence
- Behavioural finance
- The normative validity of EU theory for decision-making by a trustee
- Other measures of risk
- The application of EU theory to asset allocation
General review of the literature

- The requirement of prudence

- **Behavioural finance**

- The normative validity of EU theory for decision-making by a trustee

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General review of the literature

Tversky & Kahneman (1986):

“The … normative and the descriptive analyses of choice should be viewed as separate enterprises.”

Shiller (2003):

“… both approaches to finance, the behavioral approach, and the rational optimizing approach, have their own contributions to make …”
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Definition of the utility function

- The argument of the utility function: DC funds
- The argument of the utility function: DB funds
- Dynamic Asset Allocation
- The functional form of the utility function
- Separation of value and risk and the use of discontinuities
- Group decision-making
- Levels of risk aversion and prudence
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The argument of the utility function: the DC benefit ratio

\[ z_T = \frac{A_T \cdot a_{T-t} a_0}{A_0 \cdot a_T} \]

where:

- \( A_T \) is the member’s accumulated balance at time \( T \), being her/his retirement date;
- \( a_T \) is the price per unit, at time \( T \), of an inflation-protected immediate annuity;
- \( A_0 \) is the member’s current balance (at time 0); and
- \( a_{T-t} a_0 \) is the price per unit of an inflation-protected deferred annuity payable with effect from retirement, notionally purchased at time 0.
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The argument of the utility function: the DB benefit ratio

\[ z_t = \frac{A_t + P_t}{\dot{L}_t + \dot{P}_t} \]

where:

- \( A_t \) is the value of the assets of the fund available at time \( t \) for subsequent benefits;

- \( \dot{L}_t \) is the value of the liabilities of the fund at time \( t \) for subsequent reasonable expectations of benefits in respect of service to time 0;

- \( P_t \) is the value of payments actually made during the period \([0, t]\), accumulated to time \( t \) with interest at the risk-free rate from time to time; and

- \( \dot{P}_t \) is the value of payments of reasonable benefit expectations during the period \([0, t]\), accumulated to time \( t \) with interest at the risk-free rate from time to time.
The argument of the utility function: DB funds

\[ z_t = \frac{A_t + P_t}{L_t + \dot{P}_t} \]

\[ z_0 = \frac{A_0}{\dot{L}_0} \]

\[ z_T = \frac{A_T + P_T}{\dot{P}_T} \]
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Dynamic asset allocation
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The functional form of the utility function

- Constant absolute risk aversion:
  \[ \lambda(z) = -\frac{u''(z)}{u'(z)} = \lambda \Rightarrow u(z) = -\exp\{-\lambda z\} \]

- Constant relative risk aversion:
  \[ \gamma(z) = -z \frac{u''(z)}{u'(z)} = \lambda \Rightarrow u(z) = \frac{z^{1-\gamma} - 1}{1 - \gamma} \text{ for } \gamma > 1 \]
The functional form of the utility function

- Constant absolute risk aversion:

\[ \lambda(z) = -\frac{u''(z)}{u'(z)} = \lambda \Rightarrow u(z) = \exp\{\alpha z\} \]

- Constant relative risk aversion:

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- HARA utility function:

\[ \lambda(z) = -\frac{u''(z)}{u'(z)} = \frac{1}{a + bz} \Rightarrow u(z) = \begin{cases} 
\frac{\gamma}{1-\gamma} \left(\frac{\alpha z}{\gamma} + \beta\right)^{1-\gamma} & \text{for } b \neq 0, \gamma \neq 1; \\
\ln(z + \beta) & \text{for } b \neq 0, \gamma = 1; \\
-\exp(\alpha z) & \text{for } b = 0.
\end{cases} \]
The functional form of the utility function

\[ u(z) \]

\[ z \]
The functional form of the utility function

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- Constant or decreasing relative risk aversion:
  \[ \gamma'(z) \leq 0 \]
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Group decision-making

- Consensus:
  \[ u(z) \]

- Compromise:
  \[ u(z) = \sum_{m=1}^{M} c_m u_m(z) \]

- Equal weighting:
  \[ u(z) = \frac{1}{M} \sum_{m=1}^{M} u_m(z) \]
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Levitan & Thomson

Average relative risk aversion

Cumulative distribution
Levels of risk aversion and prudence

- Coefficient of relative risk aversion:

\[ \gamma(z) = -z \frac{u''(z)}{u'(z)} \]

- Kimball's 'coefficient of relative prudence':

\[ \pi(z) = -z \frac{u'''(z)}{u''(z)} \]
Levels of risk aversion and prudence

Criteria required:

1. coverage of the range of outcomes
2. continuity
3. unsatiation
4. relative risk aversion satisfactorily high
5. non-increasing relative risk aversion
The quantification of type-2 prudence

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4. Decreasing relative risk aversion: the WARRA-class utility function

5. Counter-intuitive results and the challenge to prudence

6. Summary
The WARRA-class utility function

\[ u(z) = \frac{u_0(z) + cu_\infty(z)}{1 + c} \]

where:

\[ u_0(z) = \frac{z^{1-\gamma_0} - 1}{1 - \gamma_0} \]

\[ u_\infty(z) = \frac{z^{1-\gamma_\infty} - 1}{1 - \gamma_\infty} \]

\[ c > 0 \]

\[ \gamma_0 \geq \gamma_\infty > 1 \]
The WARRA-class utility function

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\[ c > 0 \]

\[ \gamma_0 \geq \gamma_\infty > 1 \]
WARRA-class relative risk aversion

\[ \gamma(z) = \frac{\gamma_0 + c\gamma_\infty z^\lambda}{1 + cz^\lambda}; \]

where:

\[ \lambda = \gamma_0 - \gamma_\infty \]
WARRA-class relative risk aversion

\[ \gamma(z) \]

\( z \)

0.0 0.5 1.0 1.5 2.0

0.0 0.5 1.0 1.5 2.0

2013 Convention 31 Oct & 1 Nov
WARRA-class relative risk aversion

\[ \gamma(z) \]

\( z \)

\( \gamma(z) \) vs. \( z \)
WARRA-class relative risk aversion
WARRA-class relative risk aversion

\[ \gamma(z) \]

Trustee A
WARRA-class relative risk aversion

\[ \gamma(z) \]

Trusted A

Trusted B

2013 Convention
31 Oct & 1 Nov
Parameterising a WARRA-class utility function

\[ u(z) \]

2013 Convention
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Parameterising a WARRA-class utility function

\[ u(z) \]

\[ z \]

Graph showing the utility function \( u(z) \) over the range of \( z \) from 0.5 to 1.5.
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Illustrative results: parameters of the indirect utility functions
Illustrative results: parameters of the indirect utility functions
Illustrative results: parameters of the indirect utility functions

![Graph showing parameter values over time for different variables: gamma_0, gamma_inf, and c. The x-axis represents time (t), and the y-axis represents parameter value. The graph illustrates how these parameters change over time.]
Illustrative results: parameters of the indirect utility functions
Illustrative results: parameters of the indirect utility functions

![Diagram showing parameters gamma.0, gamma.inf, and c over time t]
Illustrative results: parameters of the indirect utility functions
Illustrative results

- Optimal exposure to the risky asset: 75.4%
Illustrative results: parameters of the model of risky assets
Illustrative results: parameters of the model of risky assets

((0.03, 0.75))
Illustrative results: parameters of the model of risky assets

(0.03, 0.75) (0.1, 0.75)
Illustrative results: parameters of the model of risky assets
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Illustrative results: parameters of the indirect utility functions
Illustrative results: parameters of the indirect utility functions

(1.0, 0.75)
Illustrative results: parameters of the indirect utility functions

\[(1.0, 0.75)\]
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Summary

Distinction between type-1 prudence and type-2 prudence:

\[ E = \int_0^\infty u(z) f(z) \, dx \]
Summary

Distinction between \textbf{type-1 prudence} and \textbf{type-2 prudence}:

\[ E = \int_{0}^{\infty} u(z) f(z) \, dx \]
Summary

Distinction between type-1 prudence and type-2 prudence

DC benefit ratio:

\[ z_T = \frac{A_T \cdot T^{-t} a_0}{A_0 \cdot a_T} \]
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Distinction between type-1 prudence and type-2 prudence

DC benefit ratio

DB benefit ratio:

\[ z_t = \frac{A_t + P_t}{L_t + \dot{P}_t} \]
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Distinction between type-1 prudence and type-2 prudence

DC benefit ratio

DB benefit ratio

Criteria required for type-2 prudence:

1) coverage
2) continuity
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4) relative risk aversion
5) non-increasing relative risk aversion
Summary

Distinction between type-1 prudence and type-2 prudence

DC benefit ratio

DB benefit ratio

Criteria required for type-2 prudence

WARRA-class utility function:

\[ u(z) = \frac{u_0(z) + c u_\infty(z)}{1 + c} \]
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Distinction between type-1 prudence and type-2 prudence
DC benefit ratio
DB benefit ratio
Criteria required for type-2 prudence
WARRA-class utility function
Method of fitting a WARRA-class utility function*

* including an algorithm & R code
Summary

Distinction between type-1 prudence and type-2 prudence

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DB benefit ratio

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WARRA-class utility function

Method of fitting a WARRA-class utility function*

Method of determining optimal exposure using dynamic asset allocation*

* including an algorithm & R code
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DC benefit ratio

DB benefit ratio

Criteria required for type-2 prudence

WARRA-class utility function

Method of fitting a WARRA-class utility function*

Method of determining optimal exposure using dynamic asset allocation*

Resolution of counter-intuitive effects previously reported

* including an algorithm & R code
Afterthought

From those in whom the workers’ trust is placed,
Prudence is their first demand;
May the stepping-stones to the shore of their dreams
Be laid with prudent hand.

Rob Thomson
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